

UNIVERSITÄT LEIPZIG
FAKULTÄT FÜR MATHEMATIK UND INFORMATIK
MATHEMATISCHES INSTITUT

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of Dieudonné-Rashevsky type

Marcus Wagner

Preprint-Reihe des Mathematischen Instituts
Preprint Nr. 02/2014

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1. The regularity result.

In [WAGNER 09] and [WAGNER 14], it has been shown that global as well as strong local minimizers (x^*, u^*) of multidimensional control problems of Dieudonné-Rashevsky type

$$F(x, u) = \int_{\Omega} f(s, x(s), u(s)) ds \longrightarrow \inf!; \quad (x, u) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm}); \quad (1.1)$$

$$Jx(s) = \begin{pmatrix} \partial x_1(s)/\partial s_1 & \dots & \partial x_1(s)/\partial s_m \\ \vdots & & \vdots \\ \partial x_n(s)/\partial s_1 & \dots & \partial x_n(s)/\partial s_m \end{pmatrix} - u(s) = \mathfrak{o}_{L^p}; \quad u(s) \in A \subset \mathbb{R}^{nm} \quad \text{for a. a. } s \in \Omega; \quad (1.2)$$

with $n, m \geq 2$, $\Omega \subset \mathbb{R}^m$, $m < p < \infty$ and a compact set $A \subset \mathbb{R}^{nm}$ with nonempty interior satisfy necessary optimality conditions in the form of Pontryagin's principle provided that the data of (1.1) – (1.2) fit into a convex or polyconvex framework, cf. [WAGNER 09], p. 549, Theorem 2.2., and [WAGNER 14], p. 9, Theorem 4.3., and p. 21, Theorem 5.4. In particular, this set of conditions contains a canonical equation in integrated form, involving multipliers $\lambda_0 \geq 0$ and $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^{nm})$, which reads as follows:

$$\lambda_0 \sum_{i=1}^n \int_{\Omega} \frac{\partial f}{\partial \xi_i}(s, x^*(s), u^*(s)) \varphi_i(s) ds + \sum_{i=1}^n \sum_{j=1}^m \int_{\Omega} \frac{\partial \varphi_i}{\partial s_j}(s) y_{i,j}^{(1)}(s) ds = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^n). \quad (1.3)$$

Inserting vector functions, which differ from $\mathfrak{o}$ in a single component only, we find that (1.3) implies n first-order PDE's

$$\langle \lambda_0 \frac{\partial f}{\partial \xi_i}(s, x^*(s), u^*(s)), \varphi \rangle = - \langle \sum_{j=1}^m \frac{\partial y_{i,j}^{(1)}}{\partial s_j}, \varphi \rangle \quad \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}), \quad 1 \leq i \leq n, \quad (1.4)$$

in which the partial derivatives of $y_{i,j}^{(1)}$ have to be understood in the sense of Schwartz distributions. With regard to applications of (1.1) – (1.2) in mathematical imaging (see [BRUNE/MAURER/WAGNER 09], [FRANEK/FRANEK/MAURER/WAGNER 12] and [WAGNER 12]), we prove a refinement of the maximum principle, claiming that the multipliers $y^{(1)}$ can be chosen from an appropriate Sobolev space rather than from $L^{p/(p-1)}(\Omega, \mathbb{R}^{nm})$, thus ensuring within (1.4) the separate existence of the generalized derivatives $\partial y_{i,j}^{(1)}/\partial s_j$ as L^r -functions. Since the divergence as their sum may be an element of an L^r -space but the summands do not (cf. [BOURDAUD/WOJCIECHOWSKI 00], p. 326), the proof of this claim is nontrivial. For simplicity, we state the theorem in a special case only, assuming dimensions $n = m = 2$, polyconvexity of the integrand f (cf. [DACOROGNA 08], p. 156 f., Definition 5.1.(iii)) and convexity of the restriction set A . The extension of the theorem and its proof to the general case covered in [WAGNER 09] and [WAGNER 14] is obvious.

Theorem 1.1. *Consider the problem (1.1) – (1.2) with $n = m = 2$ under the assumptions mentioned in [WAGNER 14], p. 18, Theorem 4.11., and choose for the polyconvex integrand $f(s, \xi, v)$ a convex representative $g(s, \xi, v, \omega_2)$ in accordance with these assumptions. Further, let $A = K \subset \mathbb{R}^4$ be a compact, convex set with $\mathfrak{o} \in \text{int}(K)$. If (x^*, u^*) is a global minimizer of the problem then there exist multipliers $\lambda_0 > 0$,*

$y^{(1)} \in W^{1,p/(p-1)}(\Omega, \mathbb{R}^4)$ and $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R})$ such that the maximum condition [WAGNER 14], p. 9, (4.6), is satisfied together with the canonical equations

$$(\mathcal{K})_1 \quad \operatorname{div} y_1^{(1)}(s) = \lambda_0 \frac{\partial g}{\partial \xi_1}(s, x^*(s), u^*(s), \det u^*(s)) \quad \text{for almost all } s \in \Omega; \quad (1.5)$$

$$(\mathcal{K})_2 \quad \operatorname{div} y_2^{(1)}(s) = \lambda_0 \frac{\partial g}{\partial \xi_2}(s, x^*(s), u^*(s), \det u^*(s)) \quad \text{for almost all } s \in \Omega. \quad (1.6)$$

2. Proof of Theorem 1.1.

Let $q_1 = p/(p-1)$ and $q_2 = p/(p-2)$. The proof of the maximum principle is based on the separation of two convex variational sets C and D within the space $\mathbb{R} \times L^p(\Omega, \mathbb{R}^4) \times L^{p/2}(\Omega, \mathbb{R})$, cf. [WAGNER 14], pp. 10 – 16. The desired gain of regularity for the multiplier $y^{(1)}$ will be obtained by replacing $L^p(\Omega, \mathbb{R}^4)$, the original target space of the state equation (1.2), by the dual space $(W^{1,q_1}(\Omega, \mathbb{R}^4))^* \hookrightarrow (L^{q_1}(\Omega, \mathbb{R}^4))^* \cong L^p(\Omega, \mathbb{R}^4)$. It turns out that the separation argument still works in this extended framework. Recall that any functional $Z \in (W^{1,q_1}(\Omega, \mathbb{R}))^*$ may be represented as

$$\langle Z, \psi \rangle_{(W^{1,q_1})^* - W^{1,q_1}} = \int_{\Omega} (Z_0 \psi + Z_1 \frac{\partial \psi}{\partial s_1} + Z_2 \frac{\partial \psi}{\partial s_2}) ds \quad \text{with } Z_0, Z_1, Z_2 \in (L^{q_1}(\Omega, \mathbb{R}))^* \cong L^p(\Omega, \mathbb{R}), \quad (2.1)$$

cf. [ADAMS/FOURNIER 07], p. 62 f., Theorem 3.9.. Consequently, $L^p(\Omega, \mathbb{R}^4)$ may be continuously imbedded into $(W^{1,q_1}(\Omega, \mathbb{R}^4))^* \cong ((L^{q_1}(\Omega, \mathbb{R}^4))^*)^3 \cong (L^p(\Omega, \mathbb{R}^4))^3$ by $z \mapsto (z, \mathbf{o}, \mathbf{o})$. Since $1 < q_1 < \infty$, $(W^{1,q_1}(\Omega, \mathbb{R}^4))^*$ is a reflexive Banach space. In relation to a global minimizer (x^*, u^*) of (1.1) – (1.2), we define the sets

$$C = \{ (\varrho, z_1, z_2) \in \mathbb{R} \times (L^p(\Omega, \mathbb{R}^4) \cap (W^{1,q_1}(\Omega, \mathbb{R}^4))^*) \times L^{p/2}(\Omega, \mathbb{R}) \quad \text{with} \quad (2.2)$$

$$\varrho = \varepsilon + D_x G(x^*, u^*, w^*)(x - x^*) + D_u G(x^*, u^*, w^*)(u - u^*) + D_w G(x^*, u^*, w^*)(w - w^*); \quad (2.3)$$

$$z_1 = Jx - Jx^* - (u - u^*); \quad z_2 = (w_2 - w_2^*) - D_u \det(u^*)(u - u^*); \quad (2.4)$$

$$\varepsilon \geq 0, \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^n), \quad u \in U, \quad w_2 \in L^{p/2}(\Omega, \mathbb{R}) \}; \quad (2.5)$$

$$\tilde{C}_\eta = \{ (\varrho, z_1, z_2) \in \mathbb{R} \times (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \times (W^{1,q_2}(\Omega, \mathbb{R}))^* \quad \text{with} \quad (2.6)$$

$$z_1 = Jx - Jx^* - (u - u^*), \quad \|z_1\|_{(W^{1,q_1})^*} \leq \eta; \quad (2.7)$$

$$z_2 = (w_2 - w_2^*) - \tilde{D}_u \det(u^*)(u - u^*), \quad \|z_2\|_{(W^{1,q_2})^*} \leq \eta; \quad (2.8)$$

$$x \in W_0^{1,p}(\Omega, \mathbb{R}^2), \quad u \in U_0 + K(\mathbf{o}, \eta) \subset (W^{1,q_1}(\Omega, \mathbb{R}^4))^*, \quad w_2 \in (W^{1,q_1}(\Omega, \mathbb{R}))^*, \quad \eta \geq 0; \quad (2.9)$$

$$\tilde{D} = \{ (\varrho, z_1, z_2) \in \mathbb{R} \times (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \times L^{p/2}(\Omega, \mathbb{R}) \quad \text{with} \quad (2.10)$$

$$\varrho < 0; \quad z_1 \in K(\mathbf{o}, \frac{1}{2} \|\varrho\|/K_0) \subset (W^{1,q_1}(\Omega, \mathbb{R}^4))^*; \quad z_2 \in K(\mathbf{o}, \frac{1}{2} \|\varrho\|/K_0) \subset L^{p/2}(\Omega, \mathbb{R}) \} \quad (2.11)$$

where

$$U = \{ z_1 \in L^p(\Omega, \mathbb{R}^4) \mid z_1(s) \in A = K \text{ for almost all } s \in \Omega \}, \quad (2.12)$$

$$U_0 = U \cap \{ z_1 \in L^p(\Omega, \mathbb{R}^4) \mid \exists x \in W_0^{1,p}(\Omega, \mathbb{R}^2) \text{ such that } z_1 = Jx \} \quad (2.13)$$

and $\tilde{D}_u \det(u^*): (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \rightarrow (W^{1,q_2}(\Omega, \mathbb{R}^4))^*$ is the natural extension of the linear, continuous Gâteaux derivative operator $D_u \det(u^*): L^p(\Omega, \mathbb{R}^4) \rightarrow L^{p/2}(\Omega, \mathbb{R})$. For given $Z \in (W^{1,q_1}(\Omega, \mathbb{R}^4))^*$, $Z \cong (Z_0, Z_1, Z_2) \in (L^p(\Omega, \mathbb{R}^4))^3$, the output of $\tilde{D}_u \det(u^*)(Z)$ acts as a linear, continuous functional on $\psi \in W^{1,q_2}(\Omega, \mathbb{R}^4)$ in the following way:

$$\begin{aligned}
\langle \tilde{D}_u T_2(u^*)(Z), \psi \rangle_{(W^{1,q_2})^* - W^{1,q_2}} &= \langle \tilde{D}_u T_2(u^*)(Z_0, Z_1, Z_2), \psi \rangle_{(W^{1,q_2})^* - W^{1,q_2}} \\
&= \int_{\Omega} \left(u_4^* Z_{0,1} \psi_1 - u_3^* Z_{0,2} \psi_2 - u_2^* Z_{0,3} \psi_3 + u_1^* Z_{0,4} \psi_4 \right) ds + \int_{\Omega} \left(u_4^* Z_{1,1} \frac{\partial \psi_1}{\partial s_1} - u_3^* Z_{1,2} \frac{\partial \psi_2}{\partial s_1} \right. \\
&\quad \left. - u_2^* Z_{1,3} \frac{\partial \psi_3}{\partial s_1} + u_1^* Z_{1,4} \frac{\partial \psi_4}{\partial s_1} \right) ds + \int_{\Omega} \left(u_4^* Z_{2,1} \frac{\partial \psi_1}{\partial s_2} - u_3^* Z_{2,2} \frac{\partial \psi_2}{\partial s_2} - u_2^* Z_{2,3} \frac{\partial \psi_3}{\partial s_2} + u_1^* Z_{2,4} \frac{\partial \psi_4}{\partial s_2} \right) ds.
\end{aligned} \tag{2.14}$$

$K_0 > 0$ is chosen according to the following assertion (ii). It holds still true that (i) C and $\tilde{D}$ are nonempty, convex variational sets with $\text{int}(D) \neq \emptyset$, cf. [WAGNER 14], p. 10, Proposition 4.5.; (ii) $(\varrho, z_1, z_2) \in C \cap \tilde{C}_\eta$ implies that $\varrho \geq -K_0 \eta$ where $K_0 > 0$ is a constant independent of η , cf. [WAGNER 14], p. 11, Proposition 4.7., and (iii) $\tilde{D}$ is a subset of $\tilde{C}_\eta$ with $\eta = \frac{1}{2} |\varrho|/K_0$, cf. [WAGNER 14], p. 11, Proposition 4.6.

Consequently, $C \cap \tilde{D} = \emptyset$, and C and $\tilde{D}$ may be weakly separated within the space $\mathbb{R} \times (W^{1,q_1}(\Omega, \mathbb{R}^4))^* \times L^{p/2}(\Omega, \mathbb{R})$ by a nontrivial linear, continuous functional $(\lambda_0, y^{(1)}, y^{(2)})$. Thus $y^{(1)}$ gains the claimed Sobolev regularity. ■

Remark 2.1. The nonexistence case from [BOURDAUD/WOJCIECHOWSKI 00] cannot occur since the assumed growth condition [WAGNER 14], p. 17, (4.96), guarantees that $\partial g(s, x^*, u^*, \det u^*)/\partial \xi_i \in L^{p/(p-1)}(\Omega, \mathbb{R})$, $1 \leq i \leq 2$, with $1 < p/(p-1) < \infty$.

Acknowledgement.

The present work has been supported within the project “Relaxation theorems and necessary optimality conditions for semiconvex multidimensional control problems” by the German Research Council.

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Last modification: 26.06.2014

Author’s address. Marcus Wagner: University of Leipzig, Department of Mathematics, P. O. B. 10 09 20, D-04009 Leipzig, Germany. Homepage / e-mail: www.thecitytocome.de / marcus.wagner@math.uni-leipzig.de