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Pontryagin's principle for Dieudonné-Rashevsky type problems with polyconvex data. Revised version

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1. Introduction.

The present paper is concerned with the proof of first-order necessary optimality conditions for multidimensional control problems of Dieudonné-Rashevsky type:

$$F(x,u) \;=\; \int_\Omega f(s,x(s),u(s))\,ds \longrightarrow \inf!\,; \quad (x,u) \in W_0^{1,p}(\Omega,\mathbb{R}^n) \,\times\, L^p(\Omega,\mathbb{R}^{nm})\,; \tag{1.1}$$

$$Jx(s) \;=\; \begin{pmatrix} \partial x_1(s)/\partial s_1 & \dots & \partial x_1(s)/\partial s_m \\ \vdots & & \vdots \\ \partial x_n(s)/\partial s_1 & \dots & \partial x_n(s)/\partial s_m \end{pmatrix} \;=\; u(s) \quad \text{for almost all } s \in \Omega\,; \tag{1.2}$$

$$u(s) \in \mathrm{A} \subset \mathbb{R}^{nm} \quad \text{for almost all } s \in \Omega \tag{1.3}$$

with $n,\, m \geqslant 2$, $\Omega \subset \mathbb{R}^m$, $m < p < \infty$ and a compact set $\mathrm{A} \subset \mathbb{R}^{nm}$ with nonempty interior. In the case of a convex integrand $f(s,\xi,\,\cdot\,)$ and a convex restriction set $\mathrm{A} = \mathrm{K}$, the global minimizers of $(1.1) - (1.3)$ satisfy optimality conditions in the form of Pontryagin's principle [01)] even though the usual regularity condition for the equality operator (1.2) fails. [02)] The question arises whether the Pontryagin principle and its proof can be extended to situations where the usual convexity of the data is replaced by generalized convexity notions. An answer to this question is of conceptual interest since the classical proof of the Pontryagin principle is based on an implicit convexification of the integrand as well as of the set of feasible controls. [03)]
Within the hierarchy of the generalized convexity notions, [04)] *polyconvexity* is the closest one to usual convexity. In short, a polyconvex function arises as a composition of the vector of all minors of a matricial argument with a convex function. Appearing e. g. in problems from material science, [05)] hydrodynamics [06)] and mathematical image processing, [07)] objectives with polyconvex integrands are of considerable practical importance. In the present paper, it will be shown that the proof of Pontryagin's principle for the problem $(1.1) - (1.3)$ can be maintained if the integrand $f(s,\xi,v)$ is polyconvex with respect to v while the control restriction set $\mathrm{A} = \mathrm{K}$ is still convex (Theorems 4.3., 4.4. and 4.11.). To the best of the author's knowledge, a proof of optimality conditions, which makes explicit use of the polyconvex structure of the integrand, is still missing in the literature.
A next step comprises the incorporation of *polyconvex gradient restrictions* into the proof scheme. Such restrictions frequently originate from volumetric constraints. A nice illustration is given if the function

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01) [Wagner 09], p. 549 f., Theorems 2.2. and 2.3.

02) Cf. [Ioffe/Tichomirow 79], p. 73 f., Theorem 3, Assumption c), and [Ito/Kunisch 08], p. 5 f.

03) See [Ginsburg/Ioffe 96], p. 92, Definition 3.2., and p. 96, Theorem 3.6. ("local relaxability" of the problem), as well as [Ioffe/Tichomirow 79], pp. 201 ff.

04) [Dacorogna 08], p. 156 f.

05) [Lubkoll/Schiela/Weiser 12], p. 12 f. (deformation of a compressible Ogden-type material).

06) [Kunisch/Vexler 07], p. 1371, (1.9), and p. 1376 f., (2.8) (vortex reduction for instationary flows).

07) [Burger/Modersitzki/Ruthotto 13], [Droske/Rumpf 04] and [Wagner 10], p. 5, (2.16) (hyperelastic image registration).

$x \in W^{1,\infty}(\Omega, \Omega)$ within the transformation formula for multiple integrals [08)]

$$\int_{\Omega} I(s)\, ds \;=\; \int_{\Omega} I\big(\,x(s)\,\big) \cdot \big|\det Jx(s)\,\big|\, ds \tag{1.5}$$

is considered as an unknown. [09)] In order to keep the formula applicable, we must ensure that $\det Jx(s) \neq 0$ a. e. Consequently, we obtain a polyconvex gradient restriction for x, e. g. $|\det Jx(s)\,| > 0$ or $\det Jx(s) > 0$. In the literature, an explicit statement of polyconvex restrictions is often avoided. Instead, the objectives are augmented with corresponding regularization terms. [10)] Assuming that the restriction set $\mathrm{A} = \mathrm{K} \cap \mathrm{P}$ can be represented as the intersection of a compact convex set K with nonempty interior and a polyconvex set P, we may introduce an *exact penalty term corresponding to* P. Then the proof technique works as in the former case, and we obtain a set of necessary optimality conditions in the form of Pontryagin's principle again.

The structure of the paper is as follows: After closing this section with some remarks about notation, we turn in *Section 2* to the description of polyconvexity. In *Section 3*, we state the control problem, collect all basic assumptions and provide existence theorems for global minimizers (Theorems 3.2., 2) and 3.5., 2)). In *Section 4*, Pontryagin's principle is derived in the case of a polyconvex integrand and a convex gradient restriction. We start with the formulation of the theorems in the special case of dimensions $n = m = 2$. Then we state and prove the first-order necessary optimality conditions in full generality as our main result (Theorem 4.3.) and provide an a. e. pointwise reformulation of the maximum condition (Theorem 4.4.). In *Section 5*, we describe how the theorems and the proof scheme can be carried over to control problems with polyconvex gradient restrictions (Theorems 5.4. and 5.6.). In the final *Section 6*, we outline an application of our theorems to a problem of hyperelastic image registration. The revised version of the paper combines and replaces the separate preprints [Wagner 13a] and [Wagner 13b] .

Notations.

Let $\Omega \subset \mathbb{R}^m$ be the closure of a bounded Lipschitz domain (in strong sense). Then $C^k(\Omega, \mathbb{R}^r)$ denotes the space of r-dimensional vector functions $f:\, \Omega \to \mathbb{R}^r$, whose components are continuous ($k = 0$) or k-times continuously differentiable ($k = 1, \,\dots\,, \infty$), respectively; $L^p(\Omega, \mathbb{R}^r)$ denotes the space of r-dimensional vector functions $f:\, \Omega \to \mathbb{R}^r$, whose components are integrable in the pth power ($1 \leqslant p < \infty$) or are measurable and esentially bounded ($p = \infty$). $W_0^{1,p}(\Omega, \mathbb{R}^r)$ denotes the Sobolev space of r-dimensional vector functions $f:\, \Omega \to \mathbb{R}^r$ with compactly supported components, possessing first-order weak partial derivatives and belonging together with them to the space $L^p(\Omega, \mathbb{R})$ ($1 \leqslant p < \infty$). $W_0^{1,\infty}(\Omega, \mathbb{R}^r)$ is understood as the Sobolev space of all r-vector functions $f:\, \Omega \to \mathbb{R}^r$ with Lipschitz continuous components and boundary values zero. [11)] Jx denotes the Jacobi matrix of the vector function $x \in W_0^{1,p}(\Omega, \mathbb{R}^r)$. The abbreviation "$(\forall)\, s \in \mathrm{A}$" has to be read as "for almost all $s \in \mathrm{A}$" or "for all $s \in \mathrm{A}$ except a Lebesgue null set". Finally, the symbol $\mathfrak{o}$ denotes, depending on the context, the zero element or the zero function of the underlying space. The notion of a polyconvex function will be precisely stated in the following section.

2. Polyconvex functions and polyconvex sets.

In order to describe polyconvexity, we introduce first the following notation for the vector of the minors of a matricial argument. [12)]

[08)] [Elstrodt 96] , p. 208, Corollary 4.9.

[09)] Cf. also [Pedregal 08] .

[10)] See e. g. the discussion of the hyperelastic registration problem from [Burger/Modersitzki/Ruthotto 13] in Section 6 below.

[11)] [Evans/Gariepy 92] , p. 131, Theorem 5.

[12)] For all notations related to matricial arguments and polyconvexity, we adopt the conventions from [Dacorogna 08] .

Definition 2.1. (The operator T) *Let $n,\, m \geqslant 1$ and denote* $\mathrm{Min}(n,m) = n \wedge m$.

1) We consider elements $v \in \mathbb{R}^{nm}$ as (n,m)-matrices and define $T(v) = \big(v,\, T_2 v,\, T_3 v\,,\, \dots,\, T_{(n\wedge m)} v\big) \in \mathbb{R}^{\tau(n,m)} = \mathbb{R}^{\sigma(1)} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \times \mathbb{R}^{\sigma(n\wedge m)}$ as the row vector consisting of all minors of v: $T_2 v = \mathrm{adj}_2 v$, $T_3 v = \mathrm{adj}_3 v,\ \dots\ ,\, T_{(n\wedge m)} v = \mathrm{adj}_{(n\wedge m)} v$. Consequently, we have $\sigma(k) = \binom{n}{k} \cdot \binom{m}{k}$, $1 \leqslant k \leqslant n \wedge m$. The sum of the dimensions is denoted by $\tau(n,m) = \sigma(1) + \dots + \sigma(n \wedge m)$.

2) Let $(m \wedge n) \leqslant p \leqslant \infty$. We consider elements $u \in L^p(\Omega, \mathbb{R}^{nm})$ as (n,m)-matrix functions and define the operator $T\colon\ L^p(\Omega, \mathbb{R}^{nm}) \to L^p(\Omega, \mathbb{R}^{\sigma(1)}\,) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ by $u \longmapsto Tu = \big(u,\, T_2 u,\, T_3 u\,,\, \dots,\, T_{(n\wedge m)} u\big)$ with $T_2 u = \mathrm{adj}_2 u$, $T_3 u = \mathrm{adj}_3 u,\ \dots\ ,\, T_{(n\wedge m)} u = \mathrm{adj}_{(n\wedge m)} u$.

Now we may state the definition of a polyconvex function.

Definition 2.2. (Polyconvex function) [13] *Consider elements $v \in \mathbb{R}^{nm}$ as (n,m)-matrices and elements $V \in \mathbb{R}^{\tau(n,m)}$ as row vectors. A function $f(v)\colon\ \mathbb{R}^{nm} \to \mathbb{R} \cup \{\,(+\infty)\,\}$ is called polyconvex iff there exists a convex function $g(V)\colon\ \mathbb{R}^{\tau(n,m)} \to \mathbb{R} \cup \{\,(+\infty)\,\}$ such that $f(v) = g\big(T(v)\big) \quad \forall\, v \in \mathbb{R}^{nm}$. The function g is called a convex representative for the polyconvex function f.*

Note that a polyconvex function may possess more than one convex representative. To a given polyconvex function f, we may associate the special convex representative [14]

$$g(V) \,=\, \inf\Big\{ \sum_{r=1}^{\tau(n,m)+1} \lambda_r\, f(v_r)\ \Big|\ \sum_{r=1}^{\tau(n,m)+1} \lambda_r\, T(v_r) \,=\, V\,,\ \sum_{r=1}^{\tau(n,m)+1} \lambda_r = 1\,,\ \lambda_r \geqslant 0\,,\ v_r \in \mathbb{R}^{nm}\,, \quad 1 \leqslant r \leqslant \tau(n,m)+1\,\Big\}\,, \tag{2.2}$$

which is called the Busemann representative of f. [15] Any polyconvex function is locally Lipschitz continuous on the interior of its effective domain [16] and, consequently, differentiable a. e. on $\mathrm{dom}\,(f)$. However, stronger smoothness properties as continuous differentiability are not automatically inherited by its convex representatives. [17] For the purposes of optimal control, it is therefore advisable to state the smoothness and growth assumptions about the integrand in terms of a fixed convex representative g rather than of the original function f.

In the special case $n = m = 2$, we get $\sigma(1) = 4$, $\sigma(2) = 1$, $\tau(2,2) = 5$ and $T(v) = \binom{v}{\det v}$. Consequently, any polyconvex function $f\colon\ \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{\,(+\infty)\,\}$ must take the form $f(v) = g\big(v,\, \det v\big)$ with a convex function $g\colon\ \mathbb{R}^5 \to \mathbb{R} \cup \{\,(+\infty)\,\}$. For $n = m = 3$, we have $\sigma(1) = 9$, $\sigma(2) = 9$, $\sigma(3) = 1$ and $\tau(3,3) = 19$. Here $\mathrm{adj}_2\, v$ is the transpose of the cofactor matrix and $\mathrm{adj}_3\, v = \det v$.

Definition 2.3. (Polyconvex set) [18] *Consider elements $v \in \mathbb{R}^{nm}$ as (n,m)-matrices. A set $\mathrm{P} \subseteq \mathbb{R}^{nm}$ is called polyconvex iff there exists a convex set $\mathrm{Q} \subseteq \mathbb{R}^{\tau(n,m)}$ such that $\mathrm{P} = \big\{\, v \in \mathbb{R}^{nm}\ \big|\ T(v) \in \mathrm{Q}\,\big\}$. The set Q is called a convex representative for the polyconvex set P.*

Equivalently, a set $\mathrm{P} \subseteq \mathbb{R}^{nm}$ can be defined as polyconvex iff its indicator function $\chi_{\mathrm{P}}\colon\ \mathbb{R}^{nm} \to \mathbb{R} \cup \{\,(+\infty)\,\}$ is a polyconvex function. [19] Elementary examples of polyconvex sets are quasiaffine hyperplanes $\mathrm{H} = \{\, v \in$

13) [Dacorogna 08] , p. 156 f., Definition 5.1.(iii).

14) [Dacorogna 08] , p. 163, Theorem 5.6., Part 2.

15) [Bevan 06] , p. 24, Definition 2.1. Recently, [Eneya/Bosse/Griewank 13] provided an effective numerical procedure for the evaluation of $g(V)$.

16) [Dacorogna 08] , p. 47, Theorem 2.31.

17) Cf. [Bevan 03] and [Bevan 06] , pp. 44 ff., Section 5.

18) [Dacorogna 08] , p. 316, Definition 7.2. (ii). The definition goes back to [Dacorogna/Ribeiro 06] , p. 108, Definition 3.1. (ii).

19) [Dacorogna 08] , p. 318, Proposition 7.5.

$\mathbb{R}^{n\times m} \,\big|\, \langle V_0\,,\, T(v)\,\rangle = \alpha_0 \,\}$ for $V_0 \in \mathbb{R}^{\tau(n,m)}$, $\alpha_0 \in \mathbb{R}$ (e. g. the group $SO(n)$), open quasiaffine half-spaces $\mathrm{H}^+ = \{\, v \in \mathbb{R}^{n\times n} \,\big|\, \langle V_0\,,\, T(v)\,\rangle > \alpha_0 \,\}$ (e. g. the group $GL^+(n)$), polyconvex polytopes obtained as the polyconvex hull of finitely many points [20] and polyconvex polyhedral sets obtained as the intersection of finitely many affine and quasiaffine half-spaces. Any convex set is polyconvex as well. [21]

Analogously to polyconvex functions, the convex representative of a polyconvex set is not necessarily uniquely determined. By the following lemma, the smallest possible convex representative is singled out, which will be called the precise representative $\widetilde{\mathrm{Q}}$ of P.

Lemma 2.4. (Precise representative of a polyconvex set) *1)* [22] *If* $\mathrm{P} \subseteq \mathbb{R}^{nm}$ *is a polyconvex set then* $\widetilde{\mathrm{Q}} = \mathrm{co}\,\big\{\, T(v) \in \mathbb{R}^{\tau(n,m)} \,\big|\, v \in \mathrm{P}\,\big\}$ *forms a convex representative of* P.

2) For any convex representative $Q \subseteq \mathbb{R}^{\tau(n,m)}$ *of* P*, it holds that* $\widetilde{\mathrm{Q}} \subseteq \mathrm{Q}$.

Proof. The proof of Part 2) is obvious. ∎

Lemma 2.5. (Compactness of the precise representative) *If* $\mathrm{P} \subset \mathbb{R}^{nm}$ *is a compact polyconvex set then its precise convex representative* $\widetilde{\mathrm{Q}} \subset \mathbb{R}^{\tau(n,m)}$ *of* P *is compact as well.*

Proof. Consider the precise representative of P, which is given by Lemma 2.4. through $\widetilde{\mathrm{Q}} = \mathrm{co}\,\big\{\, T(v) \,\big|\, v \in \mathrm{P}\,\big\} \subseteq \mathbb{R}^{\tau(n,m)}$. First, the continuous function $T\colon\ \mathbb{R}^{nm} \to \mathbb{R}^{\tau(n,m)}$ maps the compact set P onto a compact image. Secondly, the convex hull of a compact set is compact again, cf. [SCHNEIDER 93], p. 6, Theorem 1.1.10. ∎

3. Existence of optimal solutions.

a) Statement of the control problem and basic assumptions.

We are concerned with the following multidimensional control problem of Dieudonné-Rashevsky type:

$$(\mathrm{P})_0 \quad F(x,u) \,=\, \int_\Omega f(s, x(s), u(s))\, ds \;\longrightarrow\; \inf!\,; \tag{3.1}$$

$$(x,u) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm})\,; \tag{3.2}$$

$$E(x,u) \,=\, Jx(s) - u(s) \,=\, 0 \quad (\forall)\, s \in \Omega\,; \tag{3.3}$$

$$u(s) \,\in\, \mathrm{K} \cap \mathrm{P} \subset \mathbb{R}^{nm} \quad (\forall)\, s \in \Omega\,. \tag{3.4}$$

About the data within the problem $(\mathrm{P})_0$, the following assumptions will be imposed:

Assumptions 3.1. (Basic assumptions about the data within $(\mathrm{P})_0$**)**

1) We assume that $n,\, m \geqslant 2$ *and* $m < p < \infty$ *(thus* $n \wedge m < p$*).*

2) $\Omega \subset \mathbb{R}^m$ *is the closure of a bounded strongly Lipschitz domain,* $\mathrm{K} \subset \mathbb{R}^{nm}$ *is a compact convex set with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$*, and* $\mathrm{P} \subset \mathbb{R}^{nm}$ *is a nonempty compact, polyconvex set (cf. Definition 2.3. above).*

3) The integrand $f(s,\xi,v)\colon\ \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}$ *is continuous with respect to* s*,* ξ *and* v *and polyconvex as a function of* v *for all fixed* $(\hat{s}, \hat{\xi}) \in \Omega \times \mathbb{R}^n$.

4) The polyconvex integrand $f(s,\xi,v)$ *admits a convex representative* $g(s,\xi,v,\omega)\colon\ \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\,\big) \to \mathbb{R}$*, which is continuous with respect to* s *and continuously differentiable with respect to* ξ*,* v *and* ω*. Moreover,* g *satisfies a growth condition*

[20] Cf. [DACOROGNA 08] , pp. 323 ff.

[21] [DACOROGNA 08] , p. 318, Theorem 7.7.

[22] [DACOROGNA 08] , p. 317, Theorem 7.4. (iii).

$$\big|\, g(\,s,\xi,v,\omega_2,\omega_3,\,\dots,\omega_{(n\wedge m)}\,)\,\big| \,\leqslant\, A_0(s) \,+\, B_0(\xi) \,+\, C_0\,\Big(\,1+\big|\,v\,\big|^p + \sum_{r=2}^{(n\wedge m)} \big|\,\omega_r\,\big|^{p/r}\,\Big) \tag{3.5}$$
$$(\forall)\, s\in\Omega \quad \forall\,(\xi,v,\omega)\in\mathbb{R}^n\,\times\,\mathbb{R}^{nm}\times\big(\,\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\dots\,\times\mathbb{R}^{\sigma(n\wedge m)}\,\big)$$

where $A_0\in L^1(\Omega,\mathbb{R})$*,* $A_0\,\big|\,\mathrm{int}\,(\Omega)$ *is continuous,* B_0 *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$*, and* $C_0>0$*.*

b) Convex gradient restriction: existence of global minimizers.

Throughout this subsection, we assume that that $\mathrm{P}=\mathrm{K}$. Choosing for the polyconvex integrand $f(s,\xi,v)$ a convex representative $g(s,\xi,v,\omega)\colon\,\Omega\,\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\,\big(\,\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\dots\,\times\mathbb{R}^{\sigma(n\wedge m)}\,\big)\to\mathbb{R}$ according to Assumption 3.1., 4), the problem $(\mathrm{P})_0$ may be restated in the following way:

$$(\mathrm{P})_1\quad G(x,u,w)\,=\,\int_\Omega g(s,x(s),u(s),w(s))\,ds\,\longrightarrow\,\inf!\,; \tag{3.6}$$
$$(x,u,w)\in W^{1,p}_0(\Omega,\mathbb{R}^n)\times L^p(\Omega,\mathbb{R}^{nm}) \tag{3.7}$$
$$\times\,\Big(\,L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\times\dots\times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\Big)\,;$$
$$E_1(x,u)\,=\,Jx(s)-u(s)\,=\,0\quad(\forall)\,s\in\Omega\,; \tag{3.8}$$
$$E_2(u,w)\,=\,w_2(s)-\mathrm{adj}_2\,u(s)\,=\,0\quad(\forall)\,s\in\Omega\,; \tag{3.9}$$
$$E_3(u,w)\,=\,w_3(s)-\mathrm{adj}_3\,u(s)\,=\,0\quad(\forall)\,s\in\Omega\,; \tag{3.10}$$
$$\vdots$$
$$E_{(n\wedge m)}(u,w)\,=\,w_{(n\wedge m)}(s)-\mathrm{adj}_{(n\wedge m)}\,u(s)\,=\,0\quad(\forall)\,s\in\Omega\,; \tag{3.11}$$
$$u\in\mathrm{U}\,=\,\big\{\,z_1\in L^p(\Omega,\mathbb{R}^{nm})\,\big|\,z_1(s)\in\mathrm{K}\;\;(\forall)\,s\in\Omega\,\big\}\,. \tag{3.12}$$

Now we are in position to prove an existence theorem based on the equivalence of $(\mathrm{P})_0$ and $(\mathrm{P})_1$.

Theorem 3.2. (Existence of global minimizers for $(\mathrm{P})_0$**, convex gradient restriction)** *Consider problem* $(\mathrm{P})_0$ *under Assumptions 3.1. and assume further that* $\mathrm{P}=\mathrm{K}$*.*

1) If (x^*,u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then* $\big(\,x^*,\,u^*,\,T_2(u^*),\,T_3(u^*),\,\dots,\,T_{(n\wedge m)}(u^*)\,\big)$ *is a global minimizer of* $(\mathrm{P})_1$*. Conversely, if* (x^*,u^*,w^*) *is a global minimizer of* $(\mathrm{P})_1$ *then* (x^*,u^*) *is a global minimizer of* $(\mathrm{P})_0$*.*

2) There exists a global minimizer (x^*,u^*,w^*) *of* $(\mathrm{P})_1$*. Consequently, there exists a global minimizer* (x^*,u^*) *of* $(\mathrm{P})_0$ *as well.*

Proof. 1) Let (x^*,u^*) be a global minimizer of $(\mathrm{P})_0$ and assume that (x,u,w) is a feasible triple within $(\mathrm{P})_1$. Then, by definition of G, $G(x,u,w)=F(x,u)\geqslant F(x^*,u^*)=G(x^*,u^*,w^*)$ with $w^*=\big(T_2(u^*),\,T_3(u^*),\,\dots,\,T_{(n\wedge m)}(u^*)\,\big)$, and (x^*,u^*,w^*) is a global minimizer of $(\mathrm{P})_1$. On the other hand, let (x^*,u^*,w^*) be a global minimizer of $(\mathrm{P})_1$ and assume that (x,u) is feasible in $(\mathrm{P})_0$. Then, again by definition of G, we have $F(x,u)=G(x,u,w)\geqslant G(x^*,u^*,w^*)=F(x^*,u^*)$ where $w=\big(\,T_2(u),\,T_3(u),\,\dots,\,T_{(n\wedge m)}(u)\,\big)$, and (x^*,u^*) is a global minimizer of $(\mathrm{P})_0$.

2) Due to the control restriction (3.12), the feasible domain $\mathcal{B}_1$ of $(\mathrm{P})_1$ forms a bounded subset of $W^{1,p}_0(\Omega,\mathbb{R}^n)$ $\times\,L^\infty(\Omega,\mathbb{R}^{nm})\,\times\,L^\infty(\Omega,\mathbb{R}^{\sigma(2)})\,\times\,L^\infty(\Omega,\mathbb{R}^{\sigma(3)})\,\times\,\dots\,\times\,L^\infty(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$ and, consequently, of $W^{1,p}_0(\Omega,\mathbb{R}^n)$ $\times\,L^p(\Omega,\mathbb{R}^{nm})\times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\times\dots\times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$ as well. As a first consequence of the growth condition (3.5), the objective (3.6) remains bounded on $\mathcal{B}_1$, and $(\mathrm{P})_1$ admits a minimizing sequence $\{\,(x^N,u^N,w^N)\,\}$. Let us confirm that this sequence contains a subsequence, which converges with

respect to the product of the norm topology of $W^{1,p}_0(\Omega, \mathbb{R}^n)$ and the weak topologies of $L^p(\Omega, \mathbb{R}^{nm})$ and $L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ to a feasible element $(\hat{x}, \hat{u}, \hat{w})$. It is clear that we may pass over to subsequences, which satisfy $x^N \rightharpoonup \hat{x}$, $u^N \rightharpoonup \hat{u}$ and $w^N \rightharpoonup \hat{w}$ (we keep the index N). By the Rellich-Kondrachev theorem, [23] we may ensure further that x^N converges uniformly to $\hat{x}$, and $\hat{x}$ satisfies the zero boundary condition. Moreover, the weak continuity of the generalized derivative yields

$$Jx^N - u^N \;=\; E_1(x^N, u^N) \rightharpoonup E_1(\hat{x}, \hat{u}) \;=\; J\hat{x} - \hat{u} = \mathfrak{o}\,. \tag{3.13}$$

From [Dacorogna 08], p. 395 f., Theorem 8.20, Parts 3 and 4, we infer that $u^N = Jx^N \rightharpoonup J\hat{x} = \hat{u}$ implies

$$\operatorname{adj}_2 u^N \;=\; \operatorname{adj}_2 Jx^N \rightharpoonup \operatorname{adj}_2 J\hat{x} \;=\; \operatorname{adj}_2 \hat{u} \quad\Longrightarrow\quad E_2(u^N, w^N) \rightharpoonup E_2(\hat{u}, \hat{w}) \;=\; \mathfrak{o}\,; \tag{3.14}$$

$$\operatorname{adj}_3 u^N \;=\; \operatorname{adj}_3 Jx^N \rightharpoonup \operatorname{adj}_3 J\hat{x} \;=\; \operatorname{adj}_3 \hat{u} \quad\Longrightarrow\quad E_3(u^N, w^N) \rightharpoonup E_3(\hat{u}, \hat{w}) \;=\; \mathfrak{o}\,; \tag{3.15}$$

$$\vdots$$

$$\operatorname{adj}_{(n\wedge m)} u^N \;=\; \operatorname{adj}_{(n\wedge m)} Jx^N \rightharpoonup \operatorname{adj}_{(n\wedge m)} J\hat{x} \;=\; \operatorname{adj}_{(n\wedge m)} \hat{u} \tag{3.16}$$

$$\Longrightarrow\quad E_{(n\wedge m)}(u^N, w^N) \rightharpoonup E_{(n\wedge m)}(\hat{u}, \hat{w}) \;=\; \mathfrak{o}\,.$$

Note that the set U itself is convex, bounded, closed and weak*-seqentially compact as a subset of $L^p(\Omega, \mathbb{R}^{nm})$, cf. [Rolewicz 76], p. 157, Theorem IV.5.6′, and its proof. Thus $\hat{u}$ belongs to U, and $(\hat{x}, \hat{u}, \hat{w})$ is feasible in $(\mathrm{P})_1$.
As a further consequence of the growth condition (3.5), we observe that

$$\begin{aligned} \big|\, f(s, \xi, v) \,\big| \;&=\; \big|\, g(\, s, \xi, v, T_2(v), T_3(v), \ldots, T_{(n\wedge m)}(v)\,)\,\big| \\ &\leqslant\; A_0(s) \,+\, B_0(\xi) \,+\, C_0 \Big(1 + \big|\, v \,\big|^p + \sum_{r=2}^{(n\wedge m)} \big|\, T_r(v) \,\big|^{p/r} \Big) \quad (\forall)\, s \in \Omega \quad \forall\, (\xi, v) \in \mathbb{R}^n \times \mathrm{K} \end{aligned} \tag{3.17}$$

for almost all $s \in \Omega$ and arbitrary $(\xi, v) \in \mathbb{R}^n \times \mathrm{K}$ where the sum of the second and third term is a bounded function on every bounded subset of $\mathbb{R}^n \times \mathrm{K}$. Consequently, the function $\widetilde{f}(s, \xi, v) \colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R} \cup \{\, +(\infty)\,\}$ defined through

$$\widetilde{f}(s, \xi, v) \;=\; f(s, \xi, v) + \begin{cases} 0 & \big|\; (s, \xi, v) \in \Omega \times \mathbb{R}^n \times \mathrm{K}\,; \\ (+\infty) & \big|\; (s, \xi, v) \in \Omega \times \mathbb{R}^n \times (\mathbb{R}^{nm} \setminus \mathrm{K}) \end{cases} \tag{3.18}$$

belongs to the function class $\widetilde{\mathcal{F}}_{\mathrm{K}}$ described in [Wagner 11], p. 191, Definition 1.1., 2), and the existence theorems [Wagner 11], p. 193, Theorems 1.4. and 1.5., imply the weak lower semicontinuity relation

$$\liminf_{n\to\infty} G(x^N, u^N, w^N) \;=\; \liminf_{n\to\infty} \int_\Omega g(s, x^N(s), u^N(s), w^N(s))\, ds \tag{3.19}$$

$$=\; \liminf_{n\to\infty} \int_\Omega g\big(\, s, x^N(s), u^N(s), T_2\big(\, u^N(s)\,\big), T_3\big(\, u^N(s)\,\big), \ldots, T_{(n\wedge m)}\big(\, u^N(s)\,\big)\,\big)\, ds \tag{3.20}$$

$$=\; \liminf_{n\to\infty} \int_\Omega f(s, x^N(s), u^N(s))\, ds \;=\; \liminf_{n\to\infty} \int_\Omega \widetilde{f}(s, x^N(s), u^N(s)) \;\geqslant\; \int_\Omega \widetilde{f}(s, \hat{x}(s), \hat{u}(s))\, ds \tag{3.21}$$

$$=\; \int_\Omega f(s, \hat{x}(s), \hat{u}(s))\, ds \;=\; \int_\Omega g(s, \hat{x}(s), \hat{u}(s), \hat{w}(s))\, ds \;=\; G(\hat{x}, \hat{u}, \hat{w})\,. \tag{3.22}$$

Thus $(\hat{x}, \hat{u}, \hat{w})$ is a global minimizer of $(\mathrm{P})_1$. Part 1) implies now that $(\hat{x}, \hat{u})$ is a global minimizer of $(\mathrm{P})_0$. ∎

[23] [Adams/Fournier 07], p. 168, Theorem 6.3.

Remark 3.3. If the convex representative $g(s,\xi,v,\omega)$ does not depend explicitly on certain components of $\omega \in \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}$ then, obviously, the equations referring to these components may be omitted from (3.9) – (3.11), and the feasible domain of $(\mathrm{P})_1$ may be considered as a subset of an accordingly smaller space.

Remark 3.4. Theorem 3.2. remains true if Assumptions 3.1., 3) and 4) are replaced by the following weaker conditions: 3)′ $f(s,\xi,v)$ is Borel measurable with respect to s, continuous with respect to ξ and v and polyconvex as a function of v for all fixed $(\hat{s},\hat{\xi}) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n$ where $\mathrm{N} \subset \Omega$ is a m-dimensional Lebesgue null set, and 4)′ the convex representative $g(s,\xi,v,\omega)$ is Borel measurable with respect to s, continuously differentiable with respect to ξ, v and ω while still satisfying (3.5). Then the integrand f still fits into the framework described in [Wagner 11] .

c) Polyconvex gradient restriction: existence of global minimizers.

By Lemma 2.5., the compact polyconvex set P admits a convex, compact representative $\mathrm{Q} \subset \mathbb{R}^{nm} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}$. Choosing for the polyconvex integrand $f(s,\xi,v)$ a convex representative $g(s,\xi,v,\omega)$ as above, we get a reformulation $(\mathrm{Q})_1$ of problem $(\mathrm{P})_0$, which is identical with $(\mathrm{P})_1$ but comprises an addition control restriction, namely

$$
\begin{aligned}
&(\mathrm{Q})_1 \quad (3.6) \;-\; (3.12)\,;\\
&\quad (u,w) \in \mathrm{W} \;=\; \Big\{ \big(z_1, z_2, z_3, \ldots, z_{(n\wedge m)} \big) \in L^p(\Omega,\mathbb{R}^{nm}) \,\times\, L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \,\times\, L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \\
&\qquad \times \ldots \times\, L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)}) \;\Big|\; \big(z_1(s)\,,\, z_2(s)\,,\, z_3(s)\,,\, \ldots,\, z_{(n\wedge m)}(s) \big) \;\in\; \mathrm{Q} \;\; (\forall)\, s \in \Omega \Big\}\,.
\end{aligned}
\tag{3.23}
$$

Theorem 3.5. (Existence of global minimizers for $(\mathrm{P})_0$, polyconvex gradient restriction) *Consider problem $(\mathrm{P})_0$ under Assumptions 3.1.*

1) If (x^*,u^*) is a global minimizer of $(\mathrm{P})_0$ then $\big(x^*,\, u^*,\, T_2(u^*),\, T_3(u^*),\, \ldots,\, T_{(n\wedge m)}(u^*)\big)$ is a global minimizer of $(\mathrm{Q})_1$. Conversely, if (x^*,u^*,w^*) is a global minimizer of $(\mathrm{Q})_1$ then (x^*,u^*) is a global minimizer of $(\mathrm{P})_0$.

2) There exists a global minimizer (x^*,u^*,w^*) of $(\mathrm{Q})_1$ and, consequently, a global minimizer (x^*,u^*) of $(\mathrm{P})_0$.

Proof. 1) Assume that (x^*,u^*) is a global minimizer of $(\mathrm{P})_0$ and let (x,u,w) be a feasible triple within $(\mathrm{Q})_1$. Then, by definition of G, $G(x,u,w) = F(x,u) \geqslant F(x^*,u^*) = G(x^*,u^*,w^*)$ with $w^* = \big(T_2(u^*),\, T_3(u^*),\, \ldots,\, T_{(n\wedge m)}(u^*)\big)$, and (x^*,u^*,w^*) is a global minimizer of $(\mathrm{Q})_1$ as well. Conversely, if (x^*,u^*,w^*) is a global minimizer of $(\mathrm{Q})_1$ then, again by definition of G, we have $F(x,u) = G(x,u,w) \geqslant G(x^*,u^*,w^*) = F(x^*,u^*)$ for every feasible pair (x,u) within $(\mathrm{P})_0$ where $w = \big(T_2(u),\, T_3(u),\, \ldots,\, T_{(n\wedge m)}(u)\big)$. Consequently, (x^*,u^*) forms a global minimizer of $(\mathrm{P})_0$.

2) Observe that the set W defined in (3.23) is nonempty and convex together with the convex set Q. Moreover, due to the existence of a. e. pointwise convergent subsequences, the restriction $\big(z_1(s)\,,\, z_2(s)\,,\, z_3(s)\,,\, \ldots,\, z_{(n\wedge m)}(s) \big) \in \mathrm{Q}$ will be conserved under norm convergence in the space $L^p(\Omega,\mathbb{R}^{nm}) \times L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$. Consequently, W is closed in norm and weakly sequentially compact as a subset of the mentioned space. Now the same arguments as in the proof of Theorem 3.2, 2) apply. ∎

4. Pontryagin's principle: polyconvex integrand and convex gradient restrictions.

a) The special case $n = m = 2$.

First, let us illustrate the assertions of our main theorems in the simplest case with dimensions $n = m = 2$. Then a global minimizer of $(\mathrm{P})_0$ must satisfy the following first-order necessary optimality conditions.

Theorem 4.1. (Pontryagin's principle for $(\mathrm{P})_0$ with $n = m = 2$, convex restrictions) [24] *Consider the problem $(\mathrm{P})_0$ with $n = m = 2$ under Assumptions 3.1. mentioned above and choose for the polyconvex integrand $f(s,\xi,v)$ in $(\mathrm{P})_0$ a convex representative $g(s,\xi,v,\omega_2)$ in accordance with Assumption 3.1., 4). Assume further that $\mathrm{P} = \mathrm{K}$. If (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$ then there exist multipliers $\lambda_0 > 0$, $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^4)$ and $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R})$ such that the following conditions are satisfied:*

$$(\mathcal{M})\quad \lambda_0 \int_\Omega \Big(g(\,s, x^*(s), u(s), w_2(s)\,) - g(\,s, x^*(s), u^*(s), \det u^*(s)\,) \Big)\, ds - \int_\Omega \big(u(s) - u^*(s) \big)^{\mathrm{T}}\, y^{(1)}(s)\, ds \quad (4.1)$$
$$+ \int_\Omega \big(w_2(s) - \det u^*(s) \big)\, y^{(2)}(s)\, ds \; - \int_\Omega \nabla_v \det\big(u^*(s) \big)^{\mathrm{T}} \big(u(s) - u^*(s) \big)\, y^{(2)}(s)\, ds \;\geqslant\; 0$$
$$\forall\, u \in \mathrm{U} \;=\; \{\, z_1 \in L^p(\Omega, \mathbb{R}^{nm}) \;\big|\; z_1(s) \in \mathrm{K}\ (\forall)\, s \in \Omega \,\} \quad \forall\, w_2 \in L^{p/2}(\Omega, \mathbb{R})\,;$$

$$(\mathcal{K})\quad \lambda_0 \sum_{i=1}^{2} \int_\Omega \frac{\partial g}{\partial \xi_i}(\,s, x^*(s), u^*(s), \det u^*(s)\,)\,\big(x_i(s) - x_i^*(s) \big)\, ds \quad (4.2)$$
$$+ \sum_{i=1}^{2} \sum_{j=1}^{2} \int_\Omega \Big(\frac{\partial x_i}{\partial s_j}(s) - \frac{\partial x_i^*}{\partial s_j}(s) \Big)\, y_{ij}^{(1)}(s)\, ds \;=\; 0 \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,. \quad \blacksquare$$

Theorem 4.2. (Pointwise maximum condition for $(\mathrm{P})_0$ with $n = m = 2$) [25] *Consider the problem $(\mathrm{P})_0$ with $n = m = 2$ under the Assumptions 3.1. mentioned above and choose for the polyconvex integrand $f(s,\xi,v)$ in $(\mathrm{P})_0$ a convex representative $g(s,\xi,v,\omega_2)$ in accordance with Assumption 3.1., 4). Assume further that $\mathrm{P} = \mathrm{K}$. If (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$ then the maximum condition $(\mathcal{M})$ from Theorem 4.1. implies the following pointwise maximum condition:*

$$(\mathcal{MP})\quad \lambda_0 \Big(g(\,s, x^*(s), v, \omega_2\,) - g(\,s, x^*(s), u^*(s), \det u^*(s)\,) \Big) \;-\; \sum_{i=1}^{2} \sum_{j=1}^{2} \big(v_{ij} - u_{ij}^*(s) \big)\, y_{ij}^{(1)}(s) \quad (4.3)$$
$$+ \;\big(\omega_2 - \det u^*(s) \big)\, y^{(2)}(s) \;-\; \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial v_{ij}} \det(\,u^*(s)\,)\,\big(v_{ij} - u_{ij}^*(s) \big)\, y^{(2)}(s) \;\geqslant\; 0$$
$$(\forall)\, s \in \Omega \quad \forall\, v \in \mathrm{K} \quad \forall\, \omega_2 \in \mathbb{R}\,. \quad \blacksquare$$

Obviously, $(\mathcal{MP})$ can be decomposed into the separated conditions

$$(\mathcal{MP})_1\quad \lambda_0 \Big(g(\,s, x^*(s), v, \det u^*(s)\,) - g(\,s, x^*(s), u^*(s), \det u^*(s)\,) \Big) \quad (4.4)$$
$$- \sum_{i=1}^{2} \sum_{j=1}^{2} \Big(y_{ij}^{(1)}(s) + \frac{\partial}{\partial v_{ij}} \det\,(\,u^*(s)\,)\, y^{(2)}(s) \Big) \big(v_{ij} - u_{ij}^*(s) \big) \;\geqslant\; 0 \quad (\forall)\, s \in \Omega \quad \forall\, v \in \mathrm{K}\,;$$

$$(\mathcal{MP})_2\quad \lambda_0 \Big(g(\,s, x^*(s), u^*(s), \omega_2\,) - g(\,s, x^*(s), u^*(s), \det u^*(s)\,) \Big) \quad (4.5)$$
$$+ \big(\omega_2 - \det u^*(s) \big)\, y^{(2)}(s) \;\geqslant\; 0 \quad (\forall)\, s \in \Omega \quad \forall\, \omega_2 \in \mathbb{R}\,.$$

---

[24] Special case of Theorem 4.3. below.

[25] Special case of Theorem 4.4. below.

b) The main theorems in the general case $n \geqslant 2$, $m \geqslant 2$.

For general dimensions $n \geqslant 2$, $m \geqslant 2$, the first-order necessary optimality conditions for a global minimizer of the multidimensional control problem $(\mathrm{P})_0$ will be stated in the following main theorem.

Theorem 4.3. (Pontryagin's principle for $(\mathrm{P})_0$, convex restrictions) *Consider the problem $(\mathrm{P})_0$ under Assumptions 3.1. and choose for the polyconvex integrand $f(s,\xi,v)$ in $(\mathrm{P})_0$ a convex representative $g(s,\xi,v,\omega)$ in accordance with Assumption 3.1., 4). Assume further that $\mathrm{P}=\mathrm{K}$. If (x^*,u^*) is a global minimizer of $(\mathrm{P})_0$ then there exist multipliers $\lambda_0>0$, $y^{(1)}\in L^{p/(p-1)}(\Omega,\mathbb{R}^{nm})$, $y^{(2)}\in L^{p/(p-2)}(\Omega,\mathbb{R}^{\sigma(2)})$, $y^{(3)}\in L^{p/(p-3)}(\Omega,\mathbb{R}^{\sigma(3)})$, ... , $y^{(n\wedge m)}\in L^{p/(p-(n\wedge m))}(\Omega,\,\mathbb{R}^{\sigma(n\wedge m)})$ such that the following conditions are satisfied:*

$$(\mathcal{M})\quad \lambda_0\int_\Omega\Big(g(\,s,x^*(s),u(s),w(s)\,)-g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big)\,ds-\int_\Omega\big(\,u(s)-u^*(s)\,\big)^{\mathrm{T}}y^{(1)}(s)\,ds \tag{4.6}$$
$$+\sum_{r=2}^{(n\wedge m)}\int_\Omega\big(\,w_r(s)-w_r^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)\,ds-\sum_{r=2}^{(n\wedge m)}\int_\Omega\nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\,\big(\,u(s)-u^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)\,ds\;\geqslant\;0$$
$$\forall\,u\in\mathrm{U}\;=\;\big\{\,z_1\in L^p(\Omega,\mathbb{R}^{nm})\;\big|\;z_1(s)\in\mathrm{K}\;(\forall)\,s\in\Omega\,\big\}$$
$$\forall\,w_2\in L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)})\quad\forall\,w_3\in L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)})\quad\ldots\quad\forall\,w_{(n\wedge m)}\in L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\,;$$

$$(\mathcal{K})\quad\lambda_0\sum_{i=1}^n\int_\Omega\frac{\partial g}{\partial\xi_i}(\,s,x^*(s),u^*(s),w^*(s)\,)\,\big(\,x_i(s)-x_i^*(s)\,\big)\,ds \tag{4.7}$$
$$+\sum_{i=1}^n\sum_{j=1}^m\int_\Omega\Big(\frac{\partial x_i}{\partial s_j}(s)-\frac{\partial x_i^*}{\partial s_j}(s)\Big)\,y_{ij}^{(1)}(s)\,ds\;=\;0\quad\forall\,x\in W_0^{1,p}(\Omega,\mathbb{R}^n)\,.$$

Note that the regular case always occurs, i. e. $\lambda_0\neq 0$.

The following assertion shows that the condition $(\mathcal{M})$ from Theorem 4.3. implies a pointwise maximum condition.

Theorem 4.4. (Pointwise maximum condition for $(\mathrm{P})_0$) *Consider the problem $(\mathrm{P})_0$ under Assumptions 3.1. and choose for the polyconvex integrand $f(s,\xi,v)$ in $(\mathrm{P})_0$ a convex representative $g(s,\xi,v,\omega)$ in accordance with Assumption 3.1., 4). Assume further that $\mathrm{P}=\mathrm{K}$. If (x^*,u^*) is a global minimizer of $(\mathrm{P})_0$ then the maximum condition $(\mathcal{M})$ from Theorem 4.3. implies the following pointwise maximum condition:*

$$(\mathcal{MP})\quad\lambda_0\Big(\,g(\,s,x^*(s),v,\omega\,)-g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big)\;-\;\big(\,v-u^*(s)\,\big)^{\mathrm{T}}y^{(1)}(s) \tag{4.8}$$
$$+\sum_{r=2}^{(n\wedge m)}\big(\,\omega_r-w_r^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)\;-\;\sum_{r=2}^{(n\wedge m)}\nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\,\big(\,v-u^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)\;\geqslant\;0$$
$$(\forall)\,s\in\Omega\quad\forall\,v\in\mathrm{K}\quad\forall\,\omega_2\in\mathbb{R}^{\sigma(2)}\quad\forall\,\omega_3\in\mathbb{R}^{\sigma(3)}\quad\ldots\quad\forall\,\omega_{(n\wedge m)}\in\mathbb{R}^{\sigma(n\wedge m)}\,.$$

Obviously, $(\mathcal{MP})$ can be further decomposed into the following set of separated conditions:

$$(\mathcal{MP})_1\quad\lambda_0\Big(\,g(\,s,x^*(s),v,w^*(s)\,)-g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big) \tag{4.9}$$
$$-\big(\,v-u^*(s)\,\big)^{\mathrm{T}}y^{(1)}(s)\;-\;\sum_{r=2}^{(n\wedge m)}\nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\,\big(\,v-u^*(s)\,\big)^{\mathrm{T}}y^{(r)}(s)\;\geqslant\;0\quad(\forall)\,s\in\Omega\quad\forall\,v\in\mathrm{K}\,;$$

$$(\mathcal{MP})_2\quad\lambda_0\Big(\,g(\,s,x^*(s),u^*(s),\omega_2,w_3^*(s),\,\ldots\,,w_{(n\wedge m)}^*(s)\,)-g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big) \tag{4.10}$$
$$+\big(\,\omega_2-w_2^*(s)\,\big)^{\mathrm{T}}y^{(2)}(s)\;\geqslant\;0\quad(\forall)\,s\in\Omega\quad\forall\,\omega_2\in\mathbb{R}^{\sigma(2)}\,;$$

$$(\mathcal{MP})_3 \quad \lambda_0 \Big(g(\,s, x^*(s), u^*(s), w_2^*(s), \omega_3, \,\dots\,, w^*_{(n\wedge m)}(s)\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,) \Big) \tag{4.11}$$
$$+ \big(\,\omega_3 - w_3^*(s)\,\big)^{\mathrm{T}}\, y^{(3)}(s) \,\geqslant\, 0 \quad (\forall)\, s \in \Omega \quad \forall\, \omega_3 \in \mathbb{R}^{\sigma(3)}\,;$$

$$\vdots$$

$$(\mathcal{MP})_{(n\wedge m)} \quad \lambda_0 \Big(g(\,s, x^*(s), u^*(s), w_2^*(s), w_3^*(s), \,\dots\,, \omega_{(n\wedge m)}\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,) \Big) \tag{4.12}$$
$$+ \big(\,\omega_{(n\wedge m)} - w^*_{(n\wedge m)}(s)\,\big)^{\mathrm{T}}\, y^{(n\wedge m)}(s) \,\geqslant\, 0 \quad (\forall)\, s \in \Omega \quad \forall\, \omega_{(n\wedge m)} \in \mathbb{R}^{\sigma(n\wedge m)}\,.$$

c) Proof of Theorem 4.3.

Sketch of the proof. The proof of Theorem 4.3. is based on the equivalence of the problems $(\mathrm{P})_0$ and $(\mathrm{P})_1$. Thus to the given global minimizer (x^*, u^*) of $(\mathrm{P})_0$, a global minimizer (x^*, u^*, w^*) of $(\mathrm{P})_1$ corresponds, which will be used in order to define a pair of convex variational sets C and D as subsets of the space $\mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \,\dots\, \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$. We establish first that the interior of D is nonempty (Step 1). Although the usual regularity condition for the equality operator E_1 fails, [26] we are able to show that $\mathrm{C} \cap \mathrm{D} = \emptyset$ by applying Lyusternik's theorem to the operators E_2, E_3, ... , $E_{(n\wedge m)}$ (Steps $2-4$). This fact allows for the application of the weak separation theorem and a subsequent derivation of the first-order necessary optimality conditions from the resulting variational inequality (Steps 5, 6 and 7).

• **Step 1.** *The variational sets* C *and* D. Assume that a global minimizer (x^*, u^*) of $(\mathrm{P})_0$ is given. Then, by Theorem 3.2., 1), $(x^*, u^*, w^*) = \big(\,x^*, u^*, T_2(u^*), T_3(u^*), \,\dots\,, T_{(n\wedge m)}(u^*)\,\big)$ is a global minimizer of $(\mathrm{P})_1$. We define the variational sets

$$\mathrm{C} \,=\, \big\{\, \big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \dots\,,\, z_{(n\wedge m)}\,\big) \tag{4.13}$$
$$\in \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \,\dots\, \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}$$
$$\varrho = \varepsilon + D_x G(x^*, u^*, w^*)(x - x^*) + D_u G(x^*, u^*, w^*)(u - u^*) + D_w G(x^*, u^*, w^*)(w - w^*)\,; \tag{4.14}$$
$$z_1 = Jx - Jx^* - (u - u^*)\,; \tag{4.15}$$
$$z_2 = (w_2 - w_2^*) - D_u T_2(u^*)(u - u^*)\,; \tag{4.16}$$
$$\vdots$$
$$z_{(n\wedge m)} = (w_{(n\wedge m)} - w^*_{(n\wedge m)}) - D_u T_{(n\wedge m)}(u^*)(u - u^*)\,; \tag{4.17}$$
$$\varepsilon \geqslant 0\,,\; x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,, \tag{4.18}$$
$$u \in \mathrm{U}\,,\; w_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})\,,\; w_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})\,, \,\dots\,, w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,\big\}\,; \tag{4.19}$$

$$\mathrm{D} \,=\, \big\{\, \big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \dots\,,\, z_{(n\wedge m)}\,\big) \tag{4.20}$$
$$\in \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \,\dots\, \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}$$
$$\varrho < 0\,; \tag{4.21}$$
$$z_1 \in \mathrm{K}(\mathfrak{o}, \tfrac{1}{2}\,|\,\varrho\,|/K_0) \subset L^p(\Omega, \mathbb{R}^{nm})\,; \tag{4.22}$$
$$z_2 \in \mathrm{K}(\mathfrak{o}, \tfrac{1}{2}\,|\,\varrho\,|/K_0) \subset L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})\,; \tag{4.23}$$
$$\vdots$$
$$z_{(n\wedge m)} \in \mathrm{K}(\mathfrak{o}, \tfrac{1}{2}\,|\,\varrho\,|/K_0) \subset L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,\big\} \tag{4.24}$$

The value of the constant $K_0 > 0$ will be chosen according to Proposition 4.7. below.

Proposition 4.5. *The variational sets* C *and* D *are nonempty and convex with* $\operatorname{int}(\mathrm{D}) \neq \emptyset$.

[26] See [Ioffe/Tichomirow 79], p. 73 f., Theorem 3, Assumption c), and [Ito/Kunisch 08], p. 5 f.

Proof. The set C contains the origin and is convex together with U and K. Consider two points $\big(\varrho',\, z_1',\, z_2',\, z_3',\, \ldots,\, z_{(n\wedge m)}'\big)$, $\big(\varrho'',\, z_1'',\, z_2'',\, z_3'',\, \ldots,\, z_{(n\wedge m)}''\big) \in \mathrm{D}$ and parameters $\lambda',\ \lambda'' \in [\,0\,,\,1\,]$, $\lambda' + \lambda'' = 1$. It follows that $\lambda'\,\varrho' + \lambda''\,\varrho'' < 0$, and $\|\,\lambda'\,z_k' + \lambda''\,z_k''\,\|_{L^{p/k}} \leqslant \lambda'\,\|\,z_k'\,\|_{L^{p/k}} + \lambda''\,\|\,z_k''\,\|_{L^{p/k}} \leqslant \frac{1}{2}\,\big(\,\lambda'\,|\,\varrho'\,| + \lambda''\,|\,\varrho''\,|\,\big)/K_0 = \frac{1}{2}\,|\,\lambda'\,\varrho' + \lambda''\,\varrho''\,|\,/K_0$, $1 \leqslant k \leqslant n \wedge m$. Consequently, the convex combination belongs to D as well. Obviously, the point $\big(-1,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \ldots,\, \mathfrak{o}\big)$ belongs to int (D) ■

• **Step 2.** *Definition of the sets* C_η. We denote by $\mathrm{G} = \big\{\, z_1 \in L^p(\Omega, \mathbb{R}^{nm}) \,\big|\, \exists\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ such that $z_1 = Jx \,\big\}$ the subspace of the "gradients" within $L^p(\Omega, \mathbb{R}^{nm})$ and by $\mathrm{U}_0 = \mathrm{U} \cap \mathrm{G}$ the subset of those admissible controls of $(\mathrm{P})_0$, which may be completed to feasible pairs for $(\mathrm{P})_0$. For every $\eta \geqslant 0$, we define a set

$$\begin{aligned}
\mathrm{C}_\eta \;=\; \big\{\, \big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \ldots,\, z_{(n\wedge m)}\,\big) & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.25)\\
\in \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm}) \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) & \times\, \ldots\, \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \quad \text{with}\\
z_1 = Jx - Jx^* - (u - u^*)\,,\ \|\,z_1\,\|_{L^p} \leqslant \eta\,; & \qquad (4.26)\\
z_2 = (w_2 - w_2^*) - D_uT_2(u^*)(u - u^*)\,,\ \|\,z_2\,\|_{L^{p/2}} \leqslant \eta\,; & \qquad (4.27)\\
\vdots\qquad & \\
z_{(n\wedge m)} = (w_{(n\wedge m)} - w_{(n\wedge m)}^*) - D_uT_{(n\wedge m)}(u^*)(u - u^*)\,,\ \|\,z_{(n\wedge m)}\,\|_{L^{p/(n\wedge m)}} \leqslant \eta\,; & \qquad (4.28)\\
x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,,\ u \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta) \subset L^p(\Omega, \mathbb{R}^{nm})\,, & \qquad (4.29)\\
w_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})\,,\ w_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})\,,\, \ldots\,,\, w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,\big\}\,. & \qquad (4.30)
\end{aligned}$$

• **Step 3. Proposition 4.6.** *If the point* $\big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \ldots,\, z_{(n\wedge m)}\big)$ *is contained in* D *then it belongs to* C_η *with* $\eta = \frac{1}{2}\,|\,\varrho\,|/K_0$ *as well.*

Proof. Let an element $\big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \ldots,\, z_{(n\wedge m)}\big) \in \mathrm{D}$ be given. By definition, we have $\|\,z_1\,\|_{L^p} \leqslant \eta\,,\, \ldots,$ $\|\,z_{(n\wedge m)}\,\|_{L^{p/(n\wedge m)}} \leqslant \eta$ with $\eta = \frac{1}{2}\,|\,\varrho\,|/K_0$. We must confirm that the components can be represented through

$$z_1 = Jx - Jx^* - (u - u^*)\,; \tag{4.31}$$

$$z_2 = (w_2 - w_2^*) - D_uT_2(u^*)(u - u^*)\,; \tag{4.32}$$

$$\vdots$$

$$z_{(n\wedge m)} = (w_{(n\wedge m)} - w_{(n\wedge m)}^*) - D_uT_{(n\wedge m)}(u^*)(u - u^*) \tag{4.33}$$

with functions $x \in W_0^{1,p}(\Omega, \mathbb{R}^n)$, $u \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta) \subset L^p(\Omega, \mathbb{R}^{nm})$, $w_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, $w_3 \in L^{p/3}(\Omega,\, \mathbb{R}^{\sigma(3)})$, $\ldots$, $w_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$. Indeed, since $\mathfrak{o} \in \mathrm{U}_0 \subset L^p(\Omega, \mathbb{R}^{nm})$ and $\|\,z_1\,\|_{L^p} \leqslant \eta$, we may choose $x = \mathfrak{o} \in W_0^{1,p}(\Omega, \mathbb{R}^n)$, $u = \mathfrak{o} + z_1 \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta) \subset L^p(\Omega, \mathbb{R}^{nm})$, $w_2 = z_2 + D_uT_2(u^*)z_1 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, $w_3 = z_3 + D_uT_3(u^*)z_1 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})$, $\ldots$, $w_{(n\wedge m)} = z_{(n\wedge m)} + D_uT_{(n\wedge m)}(u^*)z_1 \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$, thus obtaining the claimed representation. ■

• **Step 4. Proposition 4.7.** *Let* $\eta \geqslant 0$ *be given. If* $\big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \ldots,\, z_{(n\wedge m)}\big)$ *belongs to* $\mathrm{C}_\eta \cap \mathrm{C}$ *then it follows that* $\varrho \geqslant -K_0\,\eta$ *where* $K_0 > 0$ *is a constant independent of* η.

Proof. The proof of Proposition 4.7. will be delivered in several steps.

Step 4.1. Assume that an element $\big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \ldots,\, z_{(n\wedge m)}\big)$ belongs to the intersection $\mathrm{C}_\eta \cap \mathrm{C}$. Consequently, there exist a number $\tilde{\varepsilon} \geqslant 0$ and functions $\tilde{x} \in W_0^{1,p}(\Omega, \mathbb{R}^n)$, $\tilde{u} \in \mathrm{U} \cap \big(\,\mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta)\,\big) \subset L^p(\Omega, \mathbb{R}^{nm})$, $\tilde{w}_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, $\tilde{w}_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})$, $\ldots$, $\tilde{w}_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ such that $(\tilde{u}, \tilde{w}) \in \mathrm{W}$ and

$$\varrho \;=\; \tilde{\varepsilon} + D_xG(x^*, u^*, w^*)(\tilde{x} - x^*) + D_uG(x^*, u^*, w^*)(\tilde{u} - u^*) + D_wG(x^*, u^*, w^*)(\tilde{w} - w^*)\,; \tag{4.34}$$

$$z_1 \;=\; J\tilde{x} - Jx^* - (\tilde{u} - u^*)\,; \tag{4.35}$$

$$z_2 \;=\; (\tilde{w}_2 - w_2^*) - D_uT_2(u^*)(\tilde{u} - u^*)\,; \tag{4.36}$$

$$\vdots$$

$$z_{(n\wedge m)} \;=\; (\tilde{w}_{(n\wedge m)} - w^*_{(n\wedge m)}) - D_u T_{(n\wedge m)}(u^*)(\tilde{u} - u^*) \quad \text{as well as} \tag{4.37}$$

$$\|\, z_1 \,\|_{L^p} \leqslant \eta \,, \;\|\, z_2 \,\|_{L^{p/2}} \leqslant \eta \,, \;\|\, z_3 \,\|_{L^{p/3}} \leqslant \eta \,, \;\dots ,\; \|\, z_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \leqslant \eta \,. \tag{4.38}$$

First, in relation to $\tilde{u} \in \mathrm{U}_0 + \mathrm{K}(\mathfrak{o}, \eta)$, we find $u^0 \in \mathrm{U}_0$ with $u^0 = Jx^0$, $x^0 \in W^{1,p}_0(\Omega, \mathbb{R}^n)$, and $\|\, \tilde{u} - u^0 \,\| \leqslant \eta$. [27] Thus we obtain

$$\mathfrak{o} \;=\; Jx^0 - Jx^* - (u^0 - u^*) \quad \text{and} \tag{4.39}$$

$$\|\, J\tilde{x} - Jx^0 \,\|_{L^p} \;=\; \|\, J\tilde{x} - u^0 \,\|_{L^p} \;\leqslant\; \|\, J\tilde{x} - \tilde{u} \,\|_{L^p} + \|\, \tilde{u} - u^0 \,\|_{L^p} \;\leqslant\; 2\,\eta \quad \Longrightarrow \tag{4.40}$$

$$\|\, \tilde{x} - x^0 \,\|_{W^{1,p}_0} \;\leqslant\; C_1 \,\|\, J\tilde{x} - Jx^0 \,\|_{L^p} \;\leqslant\; 2\, C_1\, \eta \tag{4.41}$$

by application of the Poincaré inequality with constant $C_1 > 0$. [28] Next, we find that

$$\tilde{w}_2 - w^*_2 \;=\; D_u T_2(u^*)(\tilde{u} - u^*) + z_2 \;=\; D_u T_2(u^*)(\tilde{u} - u^0) + D_u T_2(u^*)(u^0 - u^*) + z_2 \quad \Longrightarrow \tag{4.42}$$

$$\big(\, \tilde{w}_2 - D_u T_2(u^*)(\tilde{u} - u^0) - z_2 \,\big) - w^*_2 \;=\; D_u T_2(u^*)(u^0 - u^*)\,. \tag{4.43}$$

Using the abbreviation $\tilde{w}_2 - D_u T_2(u^*)(\tilde{u} - u^0) - z_2 = w^0_2 \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)})$, we obtain

$$\mathfrak{o} \;=\; (w^0_2 - w^*_2) - D_u T_2(u^*)(u^0 - u^*) \quad \text{and} \tag{4.44}$$

$$\|\, w^0_2 - \tilde{w}_2 \,\|_{L^{p/2}} \;\leqslant\; \|\, D_u T_2(u^*) \,\|_{\mathcal{L}(L^p, L^{p/2})} \cdot \|\, \tilde{u} - u^0 \,\|_{L^p} + \|\, z_2 \,\|_{L^{p/2}} \;\leqslant\; (1 + C_2)\, \eta\,. \tag{4.45}$$

Analogously, we find elements $w^0_3 \in L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)})$, ... , $w^0_{(n\wedge m)} \in L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ such that

$$\mathfrak{o} \;=\; (w^0_3 - w^*_3) - D_u T_3(u^*)(u^0 - u^*) \quad \text{and} \tag{4.46}$$

$$\|\, w^0_3 - \tilde{w}_3 \,\|_{L^{p/3}} \;\leqslant\; \|\, D_u T_3(u^*) \,\|_{\mathcal{L}(L^p, L^{p/3})} \cdot \|\, \tilde{u} - u^0 \,\|_{L^p} + \|\, z_3 \,\|_{L^{p/3}} \;\leqslant\; (1 + C_3)\, \eta\,; \tag{4.47}$$

$$\vdots$$

$$\mathfrak{o} \;=\; (w^0_{(n\wedge m)} - w^*_{(n\wedge m)}) - D_u T_{(n\wedge m)}(u^*)(u^0 - u^*) \quad \text{and} \tag{4.48}$$

$$\|\, w^0_{(n\wedge m)} - \tilde{w}_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \;\leqslant\; \|\, D_u T_{(n\wedge m)}(u^*) \,\|_{\mathcal{L}(L^p, L^{p/(n\wedge m)})} \cdot \|\, \tilde{u} - u^0 \,\|_{L^p} + \|\, z_{(n\wedge m)} \,\|_{L^{p/(n\wedge m)}} \;\leqslant\; (1 + C_{(n\wedge m)})\, \eta\,. \tag{4.49}$$

The constants $C_2 > 0$, $C_3 > 0$, ... , $C_{(n\wedge m)} > 0$ depend only on (x^*, u^*) and the data of $(\mathrm{P})_0$.

• **Step 4.2.** Let us invoke now Lyusternik's theorem, which reads as follows:

Theorem 4.8. (Ljusternik's theorem) [29] *Consider Banach spaces* X, Y*, the (possibly nonlinear) operator* $M\colon \mathrm{X} \to \mathrm{Y}$ *and its kernel* $\mathcal{M} = \{\, r \in \mathrm{X} \;\big|\; M(r) = \mathfrak{o} \,\}$. *If* $r^* \in \mathcal{M}$, M *is continuously Fréchet differentiable in a neighbourhood of* r^* *and* $DM(r^*)$ *maps onto* Y *then the set of the tangential vectors for* $\mathcal{M}$ *at the point* r^* *coincides with the kernel* $\{\, r \in \mathrm{X} \;\big|\; DM(r^*)(r) = \mathfrak{o} \,\}$.

Let us apply Theorem 4.8. to the data

$$\mathrm{X} \;=\; L^p(\Omega, \mathbb{R}^{nm}) \,\times\, L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \,\times\, L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \,\times\, \dots \,\times\, L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,; \tag{4.50}$$

$$\mathrm{Y} \;=\; L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \,\times\, L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \,\times\, \dots \,\times\, L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\,; \tag{4.51}$$

$$M \;=\; \big(\, E_2\,,\, \dots ,\, E_{(n\wedge m)} \,\big)\,; \tag{4.52}$$

$$r^* \;=\; (u^*, w^*)\,. \tag{4.53}$$

---

[27] If $\eta = 0$ then we may employ $u^0 = \tilde{u}$, $x^0 = \tilde{x}$ and $w^0 = \tilde{w}$ throughout the proof.

[28] [Adams/Fournier 07] , p. 184, Corollary 6.31.

[29] [Ioffe/Tichomirow 79] , p. 42.

Then we may observe that the Fréchet derivative $DM(u^*, w^*)\colon \mathrm{X} \times \mathrm{Y} \to \mathrm{Y}$, which is given through

$$DM(u^*, w^*)(\,u - u^*, w - w^*) \;=\; \begin{pmatrix} w_2 - w_2^* - D_u T_2(u^*)(u - u^*) \\ w_3 - w_3^* - D_u T_3(u^*)(u - u^*) \\ \vdots \\ w_{(n\wedge m)} - w^*_{(n\wedge m)} - D_u T_{(n\wedge m)}(u^*)(u - u^*) \end{pmatrix}, \tag{4.54}$$

is a mapping onto the target space Y. The continuity of DM with respect to the reference point is obvious. Consequently, equations (4.39), (4.44), (4.46) $-$ (4.48) imply that $(u^0 - u^*, w^0 - w^*)$ is a tangential vector for the set $\mathcal{M} = \{\,(u, w) \in \mathrm{X} \bigm| M(u, w) = \mathfrak{o}\,\}$ at (u^*, w^*), and we find elements $\big(\,Q(u^0, \lambda)\,,\; R(w^0, \lambda)\,\big) \in \mathrm{X}$ satisfying

$$\big(\,u^* + \lambda\,(u^0 - u^*) + Q(u^0, \lambda)\,,\; w^* + \lambda\,(w^0 - w^*) + R(w^0, \lambda)\,\big) \;\in\; \mathcal{M} \quad\Longleftrightarrow \tag{4.55}$$

$$w_2^*(s) + \lambda\,(w_2^0 - w_2^*) + R_2(w^0, \lambda) - \mathrm{adj}_2\,\big(\,u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0, \lambda)\,\big) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{4.56}$$

$$w_3^*(s) + \lambda\,(w_3^0 - w_3^*) + R_3(w^0, \lambda) - \mathrm{adj}_3\,\big(\,u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0, \lambda)\,\big) \;=\; 0 \quad (\forall)\, s \in \Omega\,; \tag{4.57}$$

$$\vdots$$

$$\begin{aligned} &w^*_{(n\wedge m)}(s) + \lambda\,(w^0_{(n\wedge m)} - w^*_{(n\wedge m)}) + R_{(n\wedge m)}(w^0, \lambda) \\ &\qquad - \mathrm{adj}_{(n\wedge m)}\,\big(\,u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0, \lambda)\,\big) \;=\; 0 \quad (\forall)\, s \in \Omega \end{aligned} \tag{4.58}$$

for all sufficiently small $0 \leqslant \lambda < 1$ where

$$\lim_{\lambda \to 0+0} \;\lambda^{-1}\, \|\, Q(u^0, \lambda)\,\|_{L^p} \;=\; 0\,; \tag{4.59}$$

$$\lim_{\lambda \to 0+0} \;\lambda^{-1}\, \|\, R_2(w^0, \lambda)\,\|_{L^{p/2}} \;=\; 0\,; \tag{4.60}$$

$$\vdots$$

$$\lim_{\lambda \to 0+0} \;\lambda^{-1}\, \|\, R_{(n\wedge m)}(w^0, \lambda)\,\|_{L^{p/(n\wedge m)}} \;=\; 0\,. \tag{4.61}$$

• **Step 4.3.** Let us decompose

$$\mathrm{adj}_2\,\big(\,u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0, \lambda)\,\big) \;=\; \mathrm{adj}_2\,\big(\,u^*(s) + \lambda\,(u^0 - u^*)\,\big) + S_2(u^*, u^0, \lambda)\,; \tag{4.62}$$

$$\mathrm{adj}_3\,\big(\,u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0, \lambda)\,\big) \;=\; \mathrm{adj}_3\,\big(\,u^*(s) + \lambda\,(u^0 - u^*)\,\big) + S_3(u^*, u^0, \lambda)\,; \tag{4.63}$$

$$\vdots$$

$$\mathrm{adj}_{(n\wedge m)}\,\big(\,u^*(s) + \lambda\,(u^0 - u^*) + Q(u^0, \lambda)\,\big) = \mathrm{adj}_{(n\wedge m)}\,\big(\,u^*(s) + \lambda\,(u^0 - u^*)\,\big) + S_{(n\wedge m)}(u^*, u^0, \lambda)\,. \tag{4.64}$$

Consequently, we have

$$w_2^*(s) + \lambda\,(w_2^0 - w_2^*) + R_2(w^0, \lambda) - S_2(u^*, u^0, \lambda) - \mathrm{adj}_2\,\big(\,u^*(s) + \lambda\,(u^0 - u^*)\,\big) \;=\; 0\,; \tag{4.65}$$

$$w_3^*(s) + \lambda\,(w_3^0 - w_3^*) + R_3(w^0, \lambda) - S_3(u^*, u^0, \lambda) - \mathrm{adj}_3\,\big(\,u^*(s) + \lambda\,(u^0 - u^*)\,\big) \;=\; 0\,; \tag{4.66}$$

$$\vdots$$

$$\begin{aligned} &w^*_{(n\wedge m)}(s) + \lambda\,(w^0_{(n\wedge m)} - w^*_{(n\wedge m)}) + R_{(n\wedge m)}(w^0, \lambda) - S_{(n\wedge m)}(u^*, u^0, \lambda) \\ &\qquad - \mathrm{adj}_{(n\wedge m)}\,\big(\,u^*(s) + \lambda\,(u^0 - u^*)\,\big) \;=\; 0\,, \end{aligned} \tag{4.67}$$

and the triples

$$\Big(\,x^* + \lambda\,(x^0 - x^*)\,,\; u^* + \lambda\,(u^0 - u^*)\,,\; w^* + \lambda\,(w^0 - w^*) + R(w^0, \lambda) - S(u^*, u^0, \lambda)\,\Big) \tag{4.68}$$

are feasible in $(\mathrm{P})_1$ for all sufficiently small $0 < \lambda < 1$. We will convince ourselves that the expressions $S(u^*, u^0, \lambda)$ satisfy limit relations analogous to $R(w^0, \lambda)$.

• **Step 4.4. Lemma 4.9.** *It holds that* $\lim\limits_{\lambda\to 0+0} \lambda^{-1} \| S_2(u^*, u^0, \lambda) \|_{L^{p/2}} = 0$, $\lim\limits_{\lambda\to 0+0} \lambda^{-1} \| S_3(u^*, u^0, \lambda) \|_{L^{p/3}} = 0$, ... , $\lim\limits_{\lambda\to 0+0} \lambda^{-1} \| S_{(n\wedge m)}(u^*, u^0, \lambda) \|_{L^{p/(n\wedge m)}} = 0$.

Proof. We start with expanding (4.62). Then to every index $1 \leqslant l \leqslant \sigma(2)$ correspond indices $1 \leqslant i < k \leqslant n$, $1 \leqslant j < r \leqslant m$ such that

$$S_{2,l}(u^*, u^0, \lambda) = \big(u^*_{ij} + \lambda\,(u^0_{ij} - u^*_{ij}) \big)\, Q_{kr}(u^0, \lambda) - \big(u^*_{kj} + \lambda\,(u^0_{kj} - u^*_{kj}) \big)\, Q_{ir}(u^0, \lambda) \tag{4.69}$$

$$+ Q_{ij}(u^0, \lambda)\,\big(u^*_{kr} + \lambda\,(u^0_{kr} - u^*_{kr}) \big) - Q_{kj}(u^0, \lambda)\,\big(u^*_{ir} + \lambda\,(u^0_{ir} - u^*_{ir}) \big)$$

$$+ Q_{ij}(u^0, \lambda)\, Q_{kr}(u^0, \lambda) - Q_{kj}(u^0, \lambda)\, Q_{ir}(u^0, \lambda) \quad\Longrightarrow$$

$$\int_\Omega \big| S_{2,l}(u^*, u^0, \lambda) \big|^{p/2} ds \leqslant C \Big(\int_\Omega \big| Q_{kr}(u^0, \lambda)\big|^{p/2} ds + \int_\Omega \big| Q_{ir}(u^0, \lambda) \big|^{p/2} ds + \int_\Omega \big| Q_{ij}(u^0, \lambda)\big|^{p/2} ds \tag{4.70}$$

$$+ \int_\Omega \big| Q_{kj}(u^0, \lambda)\big|^{p/2} ds + \int_\Omega \big| Q_{ij}(u^0, \lambda)\, Q_{kr}(u^0, \lambda) \big|^{p/2} ds + \int_\Omega \big| Q_{kj}(u^0, \lambda)\, Q_{ir}(u^0, \lambda) \big|^{p/2} ds \Big)$$

since $\big(u^*_{ij}(s) + \lambda\,(u^0_{ij}(s) - u^*_{ij}(s)) \big)$, $\big(u^*_{kj}(s) + \lambda\,(u^0_{kj}(s) - u^*_{kj}(s)) \big)$, $\big(u^*_{kr}(s) + \lambda\,(u^0_{kr}(s) - u^*_{kr}(s)) \big)$ and $\big(u^*_{ir}(s) + \lambda\,(u^0_{ir}(s) - u^*_{ir}(s)) \big)$ belong to the compact set K for almost all $s \in \Omega$. This implies the estimate

$$\| S_{2,l}(u^*, u^0, \lambda) \|_{L^{p/2}(\Omega)} \tag{4.71}$$

$$\leqslant C \Big(\| Q_{kr}(u^0, \lambda) \|_{L^{p/2}(\Omega)} + \| Q_{ir}(u^0, \lambda) \|_{L^{p/2}(\Omega)} + \| Q_{ij}(u^0, \lambda) \|_{L^{p/2}(\Omega)} + \| Q_{kj}(u^0, \lambda) \|_{L^{p/2}(\Omega)}$$

$$+ \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \| Q_{ir}(u^0, \lambda) \|_{L^p(\Omega)} \Big)$$

$$\leqslant \widetilde{C} \Big(\| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_r(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)} \tag{4.72}$$

$$+ \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)} + \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \| Q_{ir}(u^0, \lambda) \|_{L^p(\Omega)} \Big) \quad\Longrightarrow$$

$$\lim_{\lambda\to 0+0} \lambda^{-1} \| S_{2,l}(u^*, u^0, \lambda) \|_{L^{p/2}} \leqslant \widetilde{C} \Big(\lim_{\lambda\to 0+0} \lambda^{-1} \| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)} + \lim_{\lambda\to 0+0} \lambda^{-1} \| Q_{ir}(u^0, \lambda) \|_{L^p(\Omega)} \tag{4.73}$$

$$+ \lim_{\lambda\to 0+0} \lambda^{-1} \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} + \lim_{\lambda\to 0+0} \lambda^{-1} \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)}$$

$$+ \lim_{\lambda\to 0+0} \lambda^{-1} \| Q_{ij}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \lim_{\lambda\to 0+0} \| Q_{kr}(u^0, \lambda) \|_{L^p(\Omega)}$$

$$+ \lim_{\lambda\to 0+0} \lambda^{-1} \| Q_{kj}(u^0, \lambda) \|_{L^p(\Omega)} \cdot \lim_{\lambda\to 0+0} \| Q_{ir}(u^0, \lambda) \|_{L^p(\Omega)} \Big) = 0 \tag{4.74}$$

by assumption about $Q(u^0, \lambda)$. Analogously, the limit relations $\lambda^{-1} \| S_{3,l}(u^*, u^0, \lambda) \|_{L^{p/3}} \to 0$ for all $1 \leqslant l \leqslant \sigma(3)$, ... , $\lambda^{-1} \| S_{(n\wedge m),l}(u^*, u^0, \lambda) \|_{L^{p/(n\wedge m)}} \to 0$ for all $1 \leqslant l \leqslant \sigma(n \wedge m)$ may be confirmed. ∎

• **Step 4.5.** We compute the limit

$$\lim_{\lambda\to 0+0} \frac{1}{\lambda} \Big(G\big(x^* + \lambda\,(x^0 - x^*)\,,\, u^* + \lambda\,(u^0 - u^*)\,, \tag{4.75}$$

$$w^* + \lambda\,(w^0 - w^*) + R(w^0, \lambda) - S(u^*, u^0, \lambda) \big) - G(x^*, u^*, w^*) \Big)$$

$$= \lim_{\lambda\to 0+0} \frac{1}{\lambda} \Big(G\big(x^* + \lambda\,(x^0 - x^*)\,,\, u^* + \lambda\,(u^0 - u^*)\,,\, w^* + \lambda\,(w^0 - w^*) + R(w^0, \lambda) - S(u^*, u^0, \lambda) \big) \tag{4.76}$$

$$- G\big(x^*\,,\, u^* + \lambda\,(u^0 - u^*)\,,\, w^* + \lambda\,(w^0 - w^*) + R(w^0, \lambda) - S(u^*, u^0, \lambda) \big) \Big)$$

$$+ \lim_{\lambda\to 0+0} \frac{1}{\lambda} \Big(G\big(x^*\,,\, u^* + \lambda\,(u^0-u^*)\,,\, w^* + \lambda\,(w^0-w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda)\,\big)$$

$$- G\big(x^*\,,\, u^*\,,\, w^* + \lambda\,(w^0-w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda)\,\big)\Big)$$

$$+ \lim_{\lambda\to 0+0} \frac{1}{\lambda} \Big(G\big(x^*\,,\, u^*\,,\, w^* + \lambda\,(w^0-w^*) + R(w^0,\lambda) - S(u^*,u^0,\lambda)\,\big) - G(x^*,u^*,w^*)\Big)$$

$$= D_xG(x^*,u^*,w^*)\,(x^0-x^*) + D_uG(x^*,u^*,w^*)\,(u^0-u^*) + D_wG(x^*,u^*,w^*)\,(w^0-w^*) \geqslant 0 \tag{4.77}$$

since, by Step 4.3., the variation runs along a feasible direction. Inequality (4.77) implies that

$$D_xG(x^*,u^*,w^*)\,(\tilde{x}-x^*) + D_uG(x^*,u^*,w^*)\,(\tilde{u}-u^*) + D_wG(x^*,u^*,w^*)\,(\tilde{w}-w^*) \tag{4.78}$$

$$+ D_xG(x^*,u^*,w^*)\,(x^0-\tilde{x}) + D_uG(x^*,u^*,w^*)\,(u^0-\tilde{u}) + D_wG(x^*,u^*,w^*)\,(w^0-\tilde{w}) \geqslant 0\,.$$

By Assumption 3.1., 4), the operators $D_xG(x^*,u^*,w^*)$, $D_uG(x^*,u^*,w^*)$ and $D_wG(x^*,u^*,w^*)$ are bounded. Consequently, the first component ϱ of our element from Step 4.1. satisfies

$$\varrho = \tilde{\varepsilon} + D_xG(x^*,u^*,w^*)\,(\tilde{x}-x^*) + D_uG(x^*,u^*,w^*)\,(\tilde{u}-u^*) + D_wG(x^*,u^*,w^*)\,(\tilde{w}-w^*) \tag{4.79}$$

$$\geqslant -\Big(\big\|\, D_xG(x^*,u^*,w^*)\,\big\| \cdot \|\, x^0-\tilde{x}\,\|_{L^p} + \big\|\, D_uG(x^*,u^*,w^*)\,\big\| \cdot \|\, u^0-\tilde{u}\,\|_{L^p} \tag{4.80}$$

$$+ \big\|\, D_wG(x^*,u^*,w^*)\,\big\| \cdot \sum_{r=2}^{(n\wedge m)} \|\, w_r^0 - \tilde{w}_r\,\|_{L^{p/r}}\Big) \geqslant -C_0\Big(2\,C_1 + 1 + \sum_{r=2}^{(n\wedge m)} (1+C_r)\Big)\,\eta = -K_0\,\eta \tag{4.81}$$

where (4.41), (4.45), (4.47) and (4.49) have been used, and the proof of Proposition 4.7. is complete. ■

In particular, Proposition 4.7. implies that the origin $\big(0,\,\mathfrak{o},\,\mathfrak{o},\,\mathfrak{o},\,\dots,\,\mathfrak{o}\big)$, which belongs to $\mathrm{C}_0\cap\mathrm{C}$, must be a boundary point of C.

• **Step 5.** *Separation of* C *and* D. Propositions 4.6. and 4.7. imply together that the convex sets C and D are disjoint while $\mathrm{int}\,(\mathrm{D}) \neq \emptyset$. Indeed, by Proposition 4.6., a given point $\big(\,\varrho,\, z_1,\, z_2,\, z_3,\, \dots,\, z_{(n\wedge m)}\big) \in \mathrm{D}$ must be contained in $\mathrm{C}_{|\,\varrho\,|/(2K_0)}$. If the same point belongs to $\mathrm{C}\cap\mathrm{C}_{|\,\varrho\,|/(2K_0)}$ then we arrive at a contradiction since $\varrho < 0$ but $\varrho \geqslant -|\,\varrho\,|/2$ by Proposition 4.7. Consequently, we may apply the weak separation theorem [30] in order to find a nontrivial linear, continuous functional $(\,\lambda_0,\, y^{(1)},\, y^{(2)},\, y^{(3)},\, \dots,\, y^{(n\wedge m)}) \in \mathbb{R}\times L^{p/(p-1)}(\Omega,\mathbb{R}^{nm}) \times L^{p/(p-2)}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/(p-3)}(\Omega,\mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(p-(n\wedge m))}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$, which separates C and D properly. Consequently, we arrive at the variational inequality

$$\lambda_0\,\varrho' + \langle\, y^{(1)},\, z_1'\,\rangle + \langle\, y^{(2)},\, z_2'\,\rangle + \langle\, y^{(3)},\, z_3'\,\rangle + \dots + \langle\, y^{(n\wedge m)},\, z_{(n\wedge m)}'\,\rangle \tag{4.82}$$

$$\geqslant \lambda_0\,\varrho'' + \langle\, y^{(1)},\, z_1''\,\rangle + \langle\, y^{(2)},\, z_2''\,\rangle + \langle\, y^{(3)},\, z_3''\,\rangle + \dots + \langle\, y^{(n\wedge m)},\, z_{(n\wedge m)}''\,\rangle$$

$$\forall\,\big(\,\varrho',\, z_1',\, z_2',\, z_3',\, \dots,\, z_{(n\wedge m)}'\big) \in \mathrm{C} \quad \forall\,\big(\,\varrho'',\, z_1'',\, z_2'',\, z_3'',\, \dots,\, z_{(n\wedge m)}''\big) \quad \text{with}$$

$$\varrho'' \leqslant 0 \quad \text{and} \tag{4.83}$$

$$\|\, z_1''\,\|_{L^p} \leqslant \frac{|\,\varrho''\,|}{2\,K_0}\,,\ \|\, z_2''\,\|_{L^{p/2}} \leqslant \frac{|\,\varrho''\,|}{2\,K_0}\,,\ \|\, z_3''\,\|_{L^{p/3}} \leqslant \frac{|\,\varrho''\,|}{2\,K_0}\,,\ \dots,\ \|\, z_{(n\wedge m)}''\,\|_{L^{p/(n\wedge m)}} \leqslant \frac{|\,\varrho''\,|}{2\,K_0}\,. \tag{4.84}$$

• **Step 6.** *Derivation of the optimality conditions from the variational inequality* (4.82).

a) Nonnegativity. Inserting $(\,1,\,\mathfrak{o},\,\mathfrak{o},\,\mathfrak{o},\,\dots,\,\mathfrak{o}\,) \in \mathrm{C}$ (generated with $\varepsilon = 1$, $x = x^*$, $u = u^*$ and $w = w^*$) and $(\,-1,\,\mathfrak{o},\,\mathfrak{o},\,\mathfrak{o},\,\dots,\,\mathfrak{o}\,) \in \mathrm{D}$ into the inequality, we find $\lambda_0 \geqslant 0$.

---

[30] [IOFFE/TICHOMIROW 79], p. 152, Theorem 1.

b) Derivation of $(\mathcal{M})$. Next we insert into the inequality (4.82) elements of C generated with $\varepsilon = 0$, $x = x^*$ and functions u and w such that $u \in \mathrm{U}$ and $(u, w) \in \mathrm{W}$ together with $(\,0,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{cl}\,(\mathrm{D})$. This yields the maximum condition $(\mathcal{M})$, namely

$$\lambda_0 \Big(G(x^*, u, w) - G(x^*, u^*, w^*) \Big) - \langle\, y^{(1)},\, u - u^* \,\rangle + \sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, w_r - w_r^* \,\rangle - \sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, D_u T_r(u^*)\,(u - u^*)\,\rangle \geqslant 0\,. \tag{4.85}$$

c) Derivation of $(\mathcal{K})$. Insert now into (4.82) elements of C generated with $\varepsilon = 0$, $u = u^*$, $w = w^*$ and arbitrary $x \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ and $(\,0,\, \mathfrak{o},\, \mathfrak{o},\, \mathfrak{o},\, \dots,\, \mathfrak{o}\,) \in \mathrm{cl}\,(\mathrm{D})$. This yields

$$\lambda_0\, D_x G(x^*, u^*, w^*)\,(x - x^*) + \langle\, y^{(1)},\, Jx - Jx^* \,\rangle \;\geqslant\; 0\,. \tag{4.86}$$

Inserting further the element of C generated with $\varepsilon = 0$, $u = u^*$, $w = w^*$ and $(2x^* - x) \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ instead of x, we obtain the reverse inequality, and we arrive at the canonical equation $(\mathcal{K})$.

• **Step 7.** *Occurrence of the regular case* $\lambda_0 > 0$. Let us assume, on the contrary, that $\lambda_0 = 0$. Inserting then $u = u^*$ into the maximum condition (4.85), we get

$$\sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, w_r - w_r^* \,\rangle \;\geqslant\; 0 \tag{4.87}$$

for all $w \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ which is only possible if $y^{(2)}$, $y^{(3)}$, ... , $y^{(n\wedge m)} = \mathfrak{o}$. Further, condition $(\mathcal{K})$ reduces to

$$\langle\, y^{(1)}\,,\, Jx \,\rangle \;=\; \langle\, y^{(1)}\,,\, Jx^* \,\rangle \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,, \tag{4.88}$$

and this implies $\langle\, y^{(1)}\,,\, Jx^* \,\rangle = \langle\, y^{(1)}\,,\, u^* \,\rangle = 0$. Within the maximum condition, we obtain

$$-\;\langle\, y^{(1)}\,,\, u - u^* \,\rangle \;=\; -\,\langle\, y^{(1)}\,,\, u \,\rangle \;\geqslant\; 0 \quad \forall\, u \in \mathrm{U}\,. \tag{4.89}$$

Since $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ by assumption, U contains some $L^\infty(\Omega, \mathbb{R}^{nm})$-norm ball V, and we conclude that $\langle\, y^{(1)}\,,\, u \,\rangle = 0$ for all $u \in \mathrm{U} \cap \mathrm{V}$. Consequently, $y^{(1)}$ vanishes on all functions $z \in C_0^\infty(\Omega, \mathbb{R}^{nm}) \cap L^p(\Omega, \mathbb{R}^{nm})$ and thus on the whole space $L^p(\Omega, \mathbb{R}^{nm})$, cf. [Adams/Fournier 07], p. 38, Corollary 2.30. Summing up, we see that $\lambda_0 = 0$ implies $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, ... , $y^{(n\wedge m)} = \mathfrak{o}$, and we get a contradiction since the separating hyperplane between C and D was described by a nontrivial functional. We obtain $\lambda_0 > 0$, and the proof of Theorem 4.3. is complete. ■

d) Proof of Theorem 4.4.

Proof. The countable subset $\mathrm{K}^0 = \big(\,\mathrm{K} \cap \mathbb{Q}^{nm}\,\big) \times \mathbb{Q}^{\sigma(2)} \times \mathbb{Q}^{\sigma(3)} \times \dots \times \mathbb{Q}^{\sigma(n\wedge m)}$ lies dense in $\mathrm{K} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \times \mathbb{R}^{\sigma(n\wedge m)}$. Let us consider the null sets of the non-Lebesgue points of the integrable functions $g(\,\cdot\,, x^*(\,\cdot\,), u^*(\,\cdot\,),\, w^*(\,\cdot\,))$, $g(\,\cdot\,, x^*(\,\cdot\,), v^0, \omega^0)$, $\big(v^0 - u^*(\,\cdot\,)\big)^{\mathrm{T}} y^{(1)}(\,\cdot\,)$, $\big(\omega_r^0 - w_r^*(\,\cdot\,)\big)^{\mathrm{T}} y^{(r)}(\,\cdot\,)$, $\nabla_v\, \mathrm{adj}_r(\, u^*(\,\cdot\,)\,)\,\big(v^0 - u^*(\,\cdot\,)\big)^{\mathrm{T}} \cdot y^{(r)}(\,\cdot\,)$, $2 \leqslant r \leqslant (n \wedge m)$ for $(v^0, \omega^0) \in \mathrm{K}^0$. The countable union N of these null sets is still a null set. Since $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain, $\partial\Omega$ is a null set as well. [31] Let us

[31] [Wagner 06], p. 122, Lemma 9.2.

fix a point $s^0 \in \operatorname{int}(\Omega) \setminus \mathrm{N}$ as well as a pair $(v^0, \omega^0) \in \mathrm{K}^0$. Then a closed ball $\mathrm{B} = \mathrm{K}(s^0, \varepsilon)$ with sufficiently small radius $\varepsilon > 0$ is contained in $\operatorname{int}(\Omega)$, and the function pair (u, w) with

$$u(s) \;=\; \mathbb{1}_{\mathrm{B}}(s) \Big(\frac{\operatorname{Dist}(s, \partial \mathrm{B})}{\operatorname{Dist}(s^0, \partial B)} \cdot v^0 + \frac{\big(\operatorname{Dist}(s^0, \partial B) - \operatorname{Dist}(s, \partial B)\big)}{\operatorname{Dist}(s^0, \partial B)} \cdot u^*(s) \Big) + \mathbb{1}_{(\Omega \setminus \mathrm{B})}(s)\, u^*(s)\,; \tag{4.90}$$

$$w(s) \;=\; \mathbb{1}_{\mathrm{B}}(s) \Big(\frac{\operatorname{Dist}(s, \partial \mathrm{B})}{\operatorname{Dist}(s^0, \partial B)} \cdot \omega^0 + \frac{\big(\operatorname{Dist}(s^0, \partial B) - \operatorname{Dist}(s, \partial B)\big)}{\operatorname{Dist}(s^0, \partial B)} \cdot w^*(s) \Big) + \mathbb{1}_{(\Omega \setminus \mathrm{B})}(s)\, w^*(s) \tag{4.91}$$

belongs to $\mathrm{U} \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$. Since the functions mentioned above are continuous with respect to v and ω and $(u(s^0), w(s^0)) = (v^0, \omega^0)$, s^0 is a Lebesgue point of $g(\,\cdot\,, x^*(\,\cdot\,), u(\,\cdot\,), w(\,\cdot\,))$, $\big(u(\,\cdot\,) - u^*(\,\cdot\,)\big)^{\mathrm{T}} y^{(1)}(\,\cdot\,)$, $\big(w_r(\,\cdot\,) - w_r^*(\,\cdot\,)\big)^{\mathrm{T}} y^{(r)}(\,\cdot\,)$, $\frac{\partial}{\partial r} \operatorname{adj}_r(\,u^*(\,\cdot\,)\,)\,\big(u(\,\cdot\,) - u^*(\,\cdot\,)\big)^{\mathrm{T}}$ $\cdot\, y^{(r)}(\,\cdot\,)$, $2 \leqslant r \leqslant (n \wedge m)$, as well, and we are allowed to form the Lebesgue derivative of $(\mathcal{M})$ at the point s^0 after inserting (u, w) into the inequality.

Consider now a Vitali covering of Ω [32] and specify therein some decreasing sequence $\{\,\Omega^N\,\}$ of closed subsets of $\Omega \cap \mathrm{B}$ with $\bigcap_N \Omega^N = \{\,s^0\,\}$. Together with (u, w), all function pairs (u^N, w^N) with

$$u^N(s) \;=\; \mathbb{1}_{\Omega^N}(s)\, u(s) + \mathbb{1}_{(\Omega \setminus \Omega^N)}(s)\, u^*(s)\,; \tag{4.92}$$

$$w^N(s) \;=\; \mathbb{1}_{\Omega^N}(s)\, w(s) + \mathbb{1}_{(\Omega \setminus \Omega^N)}(s)\, w^*(s) \tag{4.93}$$

form admissible controls, and we get

$$\lim_{N\to\infty} \frac{1}{|\,\Omega^N\,|} \int_{\Omega^N} \lambda_0 \Big(g(\,s, x^*(s), u^N(s), w^N(s)\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,) \Big)\, ds \tag{4.94}$$

$$- \lim_{N\to\infty} \frac{1}{|\,\Omega^N\,|} \int_{\Omega^N} \big(u^N(s) - u^*(s) \big)^{\mathrm{T}} y^{(1)}(s)\, ds \;+\; \sum_{r=2}^{(n\wedge m)} \lim_{N\to\infty} \frac{1}{|\,\Omega^N\,|} \int_{\Omega^N} \big(w_r^N(s) - w_r^*(s) \big)^{\mathrm{T}} y^{(r)}(s)\, ds$$

$$- \sum_{r=2}^{(n\wedge m)} \lim_{N\to\infty} \frac{1}{|\,\Omega^N\,|} \int_{\Omega^N} \nabla_v \operatorname{adj}_r(\,u^*(s)\,)\, \big(u^N(s) - u^*(s) \big)^{\mathrm{T}} y^{(r)}(s)\, ds$$

$$= \lambda_0 \Big(g(\,s, x^*(s), v^0, \omega^0\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,) \Big) \;-\; \big(v^0 - u^*(s) \big)^{\mathrm{T}} y^{(1)}(s) \tag{4.95}$$

$$+ \sum_{r=2}^{(n\wedge m)} \big(\omega_r^0 - w_r^*(s) \big)^{\mathrm{T}} y^{(r)}(s) \;-\; \sum_{r=2}^{(n\wedge m)} \nabla_v \operatorname{adj}_r(\,u^*(s)\,)\, \big(v^0 - u^*(s) \big)^{\mathrm{T}} y^{(r)}(s) \;\geqslant\; 0\,.$$

This inequality holds for fixed $s^0 \in \operatorname{int}(\Omega) \setminus \mathrm{N}$ for arbitrary $(v^0, \omega^0) \in \mathrm{K}^0$. Since its left-hand side is a continuous function of (v, ω), it may be extended to the whole set $\mathrm{K} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}$, and the proof is complete. ■

e) Remarks and generalizations.

Consider an integrand $f(s, \xi, v)$ and its polyconvex representative $g(s, \xi, v, \omega)$, which satisfy the conditions 3)′ and 4)′ from Remark 3.4. instead of Assumptions 3.1., 3) and 4). Then the following growth conditions for the partial derivatives of g must be imposed.

Assumptions 4.10. (Weaker assumptions about the data of $(\mathrm{P})_0$**)** *Assume that* $g(s, \xi, v, \omega)\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}\big) \to \mathbb{R}$ *is a convex representative of the polyconvex integrand* $f(s, \xi, v)$*, which is measurable with respect to* s *and continuously differentiable with respect to* ξ*,* v *and* ω*. Let the partial derivatives of* g *satisfy the following growth conditions:*

$$\Big|\, \frac{\partial g}{\partial \xi_i}(\,s, \xi, v, \omega_2, \omega_3, \ldots, \omega_{(n\wedge m)}\,) \,\Big| \;\leqslant\; A_i(s) \;+\; B_i(\xi) \;+\; C_i \Big(1 + |\,v\,|^{p-1} + \sum_{r=2}^{(n\wedge m)} |\,\omega_r\,|^{(p-1)/r} \Big) \tag{4.96}$$

$$(\forall)\, s \in \Omega \quad \forall\, (\xi, v, \omega) \in \mathbb{R}^n \times \mathbb{R}^{nm} \times \big(\mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)} \big)$$

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[32] [Dunford/Schwartz 88], p. 212, Definition 2.

where $A_i \in L^{p/(p-1)}(\Omega, \mathbb{R})$*,* B_i *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$*, and* $C_i > 0$*,* $1 \leqslant i \leqslant n$*;*

$$\Big|\, \frac{\partial g}{\partial v_l}(\, s, \xi, v, \omega_2, \omega_3, \,\dots\,, \omega_{(n\wedge m)}\,) \,\Big| \;\leqslant\; A_l^{(1)}(s) \,+\, B_l^{(1)}(\xi) \,+\, C_l^{(1)} \Big(\, 1 + \big|\, v \,\big|^{p-1} + \sum_{r=2}^{(n\wedge m)} \big|\, \omega_r \,\big|^{(p-1)/r} \Big) \tag{4.97}$$

$$(\forall)\, s \in \Omega \quad \forall\, (\xi, v, \omega) \in \mathbb{R}^n \,\times\, \mathbb{R}^{nm} \times \big(\, \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \,\times \mathbb{R}^{\sigma(n\wedge m)} \,\big)$$

where $A_l^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R})$*,* $B_l^{(1)}$ *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$*, and* $C_l^{(1)} > 0$*,* $1 \leqslant l \leqslant \sigma(1) = nm$*;*

$$\Big|\, \frac{\partial g}{\partial \omega_{2,l}}(\, s, \xi, v, \omega_2, \omega_3, \,\dots\,, \omega_{(n\wedge m)}\,) \,\Big| \;\leqslant\; A_l^{(2)}(s) \,+\, B_l^{(2)}(\xi) \,+\, C_l^{(2)} \Big(\, 1 + \big|\, v \,\big|^{p-2} + \sum_{r=2}^{(n\wedge m)} \big|\, \omega_r \,\big|^{(p-2)/r} \Big) \tag{4.98}$$

$$(\forall)\, s \in \Omega \quad \forall\, (\xi, v, \omega) \in \mathbb{R}^n \,\times\, \mathbb{R}^{nm} \times \big(\, \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \,\times \mathbb{R}^{\sigma(n\wedge m)} \,\big)$$

where $A_l^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R})$*,* $B_l^{(2)}$ *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$*, and* $C_l^{(2)} > 0$*,* $1 \leqslant l \leqslant \sigma(2)$*;*

$$\vdots$$

$$\Big|\, \frac{\partial g}{\partial \omega_{(n\wedge m),l}}(\, s, \xi, v, \omega_2, \omega_3, \,\dots\,, \omega_{(n\wedge m)}\,) \,\Big| \;\leqslant\; A_l^{(n\wedge m)}(s) \,+\, B_l^{(n\wedge m)}(\xi) \,+\, C_l^{(n\wedge m)} \Big(\, 1 + \big|\, v \,\big|^{p-(n\wedge m)} \tag{4.99}$$

$$+ \sum_{r=2}^{(n\wedge m)} \big|\, \omega_r \,\big|^{(p-(n\wedge m))/r} \Big) \quad (\forall)\, s \in \Omega \quad \forall\, (\xi, v, \omega) \in \mathbb{R}^n \,\times\, \mathbb{R}^{nm} \times \big(\, \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \dots \,\times \mathbb{R}^{\sigma(n\wedge m)} \,\big)$$

where $A_l^{(n\wedge m)} \in L^{p/(p-(n\wedge m))}(\Omega, \mathbb{R})$*,* $B_l^{(n\wedge m)}$ *is measurable and bounded on every bounded subset of* $\mathbb{R}^n$*, and* $C_l^{(n\wedge m)} > 0$*,* $1 \leqslant l \leqslant \sigma(n \wedge m)$*.*

Theorem 4.11. *About the data of* $(\mathrm{P})_0$*, let Assumptions 3.1., 1), 2) and 5) as well as Assumptions 4.10. hold. Assume further that* $\mathrm{P} = \mathrm{K}$*. Then Theorems 4.3. and 4.4. remain true.*

Proof. An inspection of the proof of Theorem 4.3. reveals that conditions (4.96) – (4.99) are sufficient in order to ensure that $D_x G(x^*, u^*, w^*)$, $D_u G(x^*, u^*, w^*)$ and $D_w G(x^*, u^*, w^*)$ act as linear, continuous functionals on the spaces $W_0^{1,p}(\Omega, \mathbb{R}^n)$, $L^p(\Omega, \mathbb{R}^{nm})$ and $L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$, respectively. Consequently, the first variation of G can be expressed as in (4.77). ■

This generalization opens the way to the application of Pontryagin's principle to problems from mathematical image processing where, in general, the objectives depend on image data $I(s)$ being measurable and essentially bounded instead of continuous.

Remark 4.12. Remark 3.3. from above applies accordingly to Theorems 4.3., 4.4. and 4.11. Consequently, only those components of $w \in L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \dots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$, which appear explicitly within the objective of $(\mathrm{P})_1$, must be paired with multipliers and incorporated into the conditions $(\mathcal{M})$ and $(\mathcal{MP})$, respectively.

Remark 4.13. Theorems 4.3., 4.4. and 4.11. may be restated for strong local minimizers (x^*, u^*) of $(\mathrm{P})_0$, cf. [Ioffe/Tichomirow 79], p. 98 f.

Remark 4.14. With obvious adaptations, the proof of Theorem 4.3. applies to [Wagner 09], p. 549, Theorem 2.2. as well. Consequently, an error occuring in the proof of this theorem ibid., p. 552, Step 3, can be completely removed. Analogously, the proof of Theorem 4.4. applies to ibid., p. 550, Theorem 2.3., thus fixing an error in the proof of this theorem ibid., p. 553, (33).

5. Pontryagin's principle: polyconvex integrand and polyconvex gradient restrictions.

a) An exact penalty for the polyconvex control restriction.

Throughout this section, $\mathrm{P} \subset \mathbb{R}^{nm}$ is an arbitrary polyconvex set according to Assumption 3.1., 2). In order to carry over the proof scheme from Section 4, a further equivalent formulation of problem $(\mathrm{P})_0$ will be used. Namely, we will introduce an exact penalty for the control restriction (3.23), thus obtaining the problem

$$(\mathrm{Q})_2 \quad \widetilde{G}(x,u,w) \, = \, \int_\Omega g(s,x(s),u(s),w(s))\,ds + K_2 \cdot \mathrm{Dist}\,\big(\,(x,u,w)\,,\, L^p(\Omega,\mathbb{R}^n) \times \mathrm{W}\,\big) \, \longrightarrow \, \inf!\,; \tag{5.1}$$

$$(x,u,w) \in W_0^{1,p}(\Omega,\mathbb{R}^n) \times L^p(\Omega,\mathbb{R}^{nm}) \tag{5.2}$$

$$\times \Big(\, L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\Big);$$

$$E_1(x,u) \, = \, Jx(s) - u(s) \, = \, 0 \quad (\forall)\, s \in \Omega\,; \tag{5.3}$$

$$E_2(u,w) \, = \, w_2(s) - \mathrm{adj}_2\, u(s) \, = \, 0 \quad (\forall)\, s \in \Omega\,; \tag{5.4}$$

$$E_3(u,w) \, = \, w_3(s) - \mathrm{adj}_3\, u(s) \, = \, 0 \quad (\forall)\, s \in \Omega\,; \tag{5.5}$$

$$\vdots$$

$$E_{(n\wedge m)}(u,w) \, = \, w_{(n\wedge m)}(s) - \mathrm{adj}_{(n\wedge m)}\, u(s) \, = \, 0 \quad (\forall)\, s \in \Omega\,; \tag{5.6}$$

$$u \in \mathrm{U} \, = \, \big\{\, z_1 \in L^p(\Omega,\mathbb{R}^{nm}) \,\big|\, z_1(s) \in \mathrm{K} \;\; (\forall)\, s \in \Omega \,\big\}\,, \tag{5.7}$$

which turns out to be equivalent to $(\mathrm{P})_0$ and $(\mathrm{Q})_1$ provided that a sufficiently large constant $K_2 > 0$ will be chosen (see Proposition 5.3. below) and the partial derivatives of g satisfy Assumptions 4.10.
Before stating the next lemma, let us define the closed balls

$$\mathrm{K}(\mathfrak{o},R_0) \subset W_0^{1,p}(\Omega,\mathbb{R}^n) \hookrightarrow C_0^0(\Omega,\mathbb{R}^n)\,; \tag{5.8}$$

$$\mathrm{K}'(\mathfrak{o},R') \, = \, \mathrm{K}(\mathfrak{o},R') \,\times\, \mathrm{K}(\mathfrak{o},R') \,\times\, \ldots \,\times\, \mathrm{K}(\mathfrak{o},R') \tag{5.9}$$

$$\subset L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \,\ldots\, \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})$$

with the radii

$$R_0 \, = \, \sup\big\{\, \|\, x \,\|_{C_0^0(\Omega,\mathbb{R}^n)} \,\big|\, Jx \in \mathrm{U} \,\big\}\,; \tag{5.10}$$

$$R' \, = \, \underset{2\,\leqslant\, r\,\leqslant\,(n\wedge m)}{\mathrm{Max}} \quad C_r \cdot \sup\big\{\, \big|\, \big(\,\mathrm{adj}_r(v)\,\big)_l \,\big| \,\big|\, 1 \leqslant l \leqslant \sigma(r)\,,\; v \in \mathrm{K} \,\big\} \tag{5.11}$$

where the constants $C_r > 0$ are taken from the imbedding inequalities

$$\|\, z_r \,\|_{L^{p/r}} \, \leqslant \, C_r \,\|\, z_r \,\|_{L^\infty}\,,\; 2 \leqslant r \leqslant (n \wedge m)\,. \tag{5.12}$$

Lemma 5.1. *Let Assumptions 3.1., 1), 2) and 5) together with Assumptions 4.10. hold. Then the functional G within $(\mathrm{Q})_1$ satisfies the Lipschitz condition*

$$\big|\, G(x',u',w') - G(x'',u'',w'') \,\big| \, \leqslant \, K_1 \,\Big(\, \|\, x'-x'' \,\|_{L^p} + \|\, u'-u'' \,\|_{L^p} + \sum_{r=2}^{(n\wedge m)} \|\, w_r' - w_r'' \,\|_{L^{p/r}} \Big) \tag{5.13}$$

for all triples (x',u',w'), $(x'',u'',w'') \in \mathrm{K}(\mathfrak{o},R_0) \times \mathrm{U} \times \mathrm{K}'(\mathfrak{o},R') \subset \big(\, W_0^{1,p}(\Omega,\mathbb{R}^n) \,\cap\, C_0^0(\Omega,\mathbb{R}^n)\big) \times L^p(\Omega,\mathbb{R}^{nm}) \times \big(\, L^{p/2}(\Omega,\mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega,\mathbb{R}^{\sigma(3)}) \times \,\ldots\, \times L^{p/(n\wedge m)}(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\big)$.

Proof. Fix a number $\varepsilon > 0$ and consider an arbitrary pair of elements (x',u',w'), $(x'',u'',w'') \in \mathrm{K}(\mathfrak{o},R_0) \times \mathrm{U} \times \mathrm{K}'(\mathfrak{o},R')$. By convexity, this set contains the whole segment $\mathrm{S} = \big[\,(x',u',w')\,,\,(x'',u'',w'')\,\big]$. Our assumptions guarantee the Gâteaux differentiability of the functional G with respect to x, u, w_2, w_3, ... , $w_{(n\wedge m)}$ even on the larger set $\mathrm{K}(\mathfrak{o}, R_0+\varepsilon) \times \big(\mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)\big) \times \mathrm{K}'(\mathfrak{o}, R'+\varepsilon)$ and, consequently, along S. Now the mean value theorem [33] yields the estimate

$$\big|\, G(x',u',w') - G(x'',u'',w'')\,\big| \tag{5.14}$$
$$\leqslant \sup_{(\hat{x},\hat{u},\hat{w})\,\in\,\mathrm{S}} \big\|\, DG(\hat{x},\hat{u},\hat{w})\,\big\| \cdot \Big(\|\, x'-x''\,\|_{L^p} + \|\, u'-u''\,\|_{L^p} + \sum_{r=2}^{(n\wedge m)} \|\, w'_r - w''_r\,\|_{L^{p/r}} \Big)$$

where

$$\sup_{(\hat{x},\hat{u},\hat{w})\,\in\,\mathrm{S}} \big\|\, DG(\hat{x},\hat{u},\hat{w})\,\big\| \leqslant \sup_{\hat{x}\,\in\,\mathrm{K}(\mathfrak{o},R_0+\varepsilon)} \ \sup_{\hat{u}\,\in\,\mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)} \ \sup_{\hat{w}\,\in\,\mathrm{K}'(\mathfrak{o},R'+\varepsilon)} \big\|\, DG(\hat{x},\hat{u},\hat{w})\,\big\| \tag{5.15}$$

$$\leqslant \sup_{\hat{x}\,\in\,\mathrm{K}(\mathfrak{o},R_0+\varepsilon)} \ \sup_{\hat{u}\,\in\,\mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)} \ \sup_{\hat{w}\,\in\,\mathrm{K}'(\mathfrak{o},R'+\varepsilon)} C\,\Big(\sum_{i=1}^{n} \Big\|\, \frac{\partial g}{\partial \xi_i}(\hat{x},\hat{u},\hat{w})\,\Big\|_{L^{p/(p-1)}} + \sum_{i=1}^{n}\sum_{j=1}^{m} \Big\|\, \frac{\partial g}{\partial v_{ij}}(\hat{x},\hat{u},\hat{w})\,\Big\|_{L^{p/(p-1)}} \tag{5.16}$$
$$+ \sum_{l=1}^{\sigma(2)} \Big\|\, \frac{\partial g}{\partial \omega_{2,l}}(\hat{x},\hat{u},\hat{w})\,\Big\|_{L^{p/(p-2)}} + \,...\, + \sum_{l=1}^{\sigma(n\wedge m)} \Big\|\, \frac{\partial g}{\partial \omega_{(n\wedge m),l}}(\hat{x},\hat{u},\hat{w})\,\Big\|_{L^{p/(p-(n\wedge m))}} \Big)\,.$$

The suprema in (5.16) are formed over bounded function sets. Consequently, the expression in (5.16) remains finite as far as the boundedness of the Nemytskij operators $(\hat{x},\hat{u},\hat{w}) \longmapsto \partial g(\hat{x},\hat{u},\hat{w})/\partial\xi_i \in L^{p/(p-1)}$, $(\hat{x},\hat{u},\hat{w}) \longmapsto \partial g(\hat{x},\hat{u},\hat{w})/\partial v_{ij} \in L^{p/(p-1)}$, $(\hat{x},\hat{u},\hat{w}) \longmapsto \partial g(\hat{x},\hat{u},\hat{w})/\partial\omega_{2,l} \in L^{p/(p-2)}$, $(\hat{x},\hat{u},\hat{w}) \longmapsto \partial g(\hat{x},\hat{u},\hat{w})/\partial\omega_{3,l} \in L^{p/(p-3)}$, ... , $(\hat{x},\hat{u},\hat{w}) \longmapsto \partial g(\hat{x},\hat{u},\hat{w})/\ \partial\omega_{(n\wedge m),l} \in L^{p/(p-(n\wedge m))}$ can be guaranteed. However, this is implied by the growth conditions (4.96) – (4.99). For example, from (4.96) it follows that

$$\Big\|\, \frac{\partial g}{\partial \xi_i}(\hat{x},\hat{u},\hat{w})\,\Big\|_{L^{p/(p-1)}} = \int_\Omega \Big|\, \frac{\partial}{\partial \xi_i} g(s,\hat{x}(s),\hat{u}(s),\hat{w}(s))\,\Big|^{p/(p-1)} ds \tag{5.17}$$

$$\leqslant \int_\Omega \Big|\, A_i(s) + B_i\big(\,\hat{x}(s)\,\big) + C_i\,\Big(1 + \big|\,\hat{u}(s)\,\big|^{p-1} + \sum_{r=2}^{(n\wedge m)} \big|\,\hat{w}_r(s)\,\big|^{(p-1)/r} \Big) \Big|^{p/(p-1)} ds \tag{5.18}$$

$$\leqslant C \int_\Omega \Big(A_i(s)^{p/(p-1)} + B_i\big(\,\hat{x}(s)\,\big)^{p/(p-1)} + C_i\,\Big(1 + \big|\,\hat{u}(s)\,\big|^{p} + \sum_{r=2}^{(n\wedge m)} \big|\,\hat{w}_r(s)\,\big|^{p/r} \Big) \Big) ds \tag{5.19}$$

$$\leqslant C\,\Big(\|\, A_i\,\|_{L^{p/(p-1)}}^{p/(p-1)} + \big(\tilde{B}_i(R_0+\varepsilon)\big)^{p/(p-1)} + C_i\,\Big(1 + \|\,\hat{u}\,\|_{L^p}^{p} + \sum_{r=2}^{(n\wedge m)} \|\,\hat{w}_r\,\|_{L^{p/r}}^{p/r} \Big) \Big) \tag{5.20}$$

with an appropriate constant $\tilde{B}_i(R_0+\varepsilon) > 0$ such that $\|\,\hat{x}\,\|_{C^0} \leqslant R_0+\varepsilon \Longrightarrow |\,B_i(\,\hat{x}(s)\,)\,| \leqslant \tilde{B}_i(R_0+\varepsilon)$, and (5.20) remains uniformly bounded for all $(\hat{x},\hat{u},\hat{w}) \in \mathrm{K}(\mathfrak{o},R_0+\varepsilon) \times \big(\mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)\big) \times \mathrm{K}'(\mathfrak{o},R'+\varepsilon)$. For the other partial derivatives occuring in (5.16), we may reason analogously. Consequently, condition (5.13) holds true with a constant $K_1 \geqslant \sup_{(\hat{x},\hat{u},\hat{w})\,\in\,\mathrm{K}(\mathfrak{o},R_0+\varepsilon)\,\times\,(\,\mathrm{U}+\mathrm{K}(\mathfrak{o},\varepsilon)\,)\,\times\,\mathrm{K}'(\mathfrak{o},R'+\varepsilon)} \|\,DG(\hat{x},\hat{u},\hat{w})\,\|$. ∎

Remark 5.2. For the application of the mean value theorem in this proof, the Gâteaux differentiability of the functional G is required not only on the set $\mathrm{K}(\mathfrak{o},R_0) \times \mathrm{U} \times \big\{\, (\,z_2, z_3,\,...\,, z_{(n\wedge m)}\,) \;\big|\; (\,z_1,\, z_2,\, z_3,\,...\,, z_{(n\wedge m)}\,) \in \mathrm{W}\,\big\}$, which belongs in fact to the subspace $W_0^{1,\infty}(\Omega,\mathbb{R}^n) \times L^\infty(\Omega,\mathbb{R}^{nm}) \times \big(\, L^\infty(\Omega,\mathbb{R}^{\sigma(2)}) \times L^\infty(\Omega,\mathbb{R}^{\sigma(3)}) \times\, ... \times L^\infty(\Omega,\mathbb{R}^{\sigma(n\wedge m)})\,\big)$, but on an open neighbourhood of it. In order to ensure this, the growth conditions from Assumptions 4.10. must be imposed.

Proposition 5.3. (Equivalence of $(\mathrm{Q})_1$ **and** $(\mathrm{Q})_2$**)** *Let Assumptions 3.1., 1), 2) and 5) together with Assumptions 4.10. hold and fix in* (5.1) *a sufficiently large constant* $K_2 > K_1 > 0$. [34] *Then every global*

[33] [Ioffe/Tichomirow 79] , p. 40.

[34] The constant K_1 is taken from Lemma 5.1.

minimizer (x^*, u^*, w^*) *of* $(\mathrm{Q})_1$ *is a global minimizer of* $(\mathrm{Q})_2$ *as well. Conversely, every global minimizer of* $(\mathrm{Q})_2$ *is feasible in* $(\mathrm{Q})_1$ *and forms a global minimizer of* $(\mathrm{Q})_1$.

Proof. Denote the feasible domains of $(\mathrm{Q})_1$ and $(\mathrm{Q})_2$ by $\mathcal{B}_1$ and $\mathcal{B}_2$. BAssume that (x^*, u^*, w^*) is a global minimizer of $(\mathrm{Q})_1$. Let us apply [Clarke 90], p. 51 f., Proposition 2.4.3., to the following data: $\mathrm{S} = \mathcal{B}_2 \subset L^p(\Omega, \mathbb{R}^n) \times \mathrm{U} \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$, $\mathrm{C} = \mathcal{B}_1 = \mathcal{B}_2 \cap \big(L^p(\Omega, \mathbb{R}^n) \times \mathrm{W} \big)$, and $f\colon \mathrm{S} \to \mathbb{R}$ is the functional $G\colon \mathcal{B}_2 \to \mathbb{R}$. By Lemma 5.1., G is Lipschitz on S with constant K_1, and from the proof of Theorem 3.5., 2), we see that W is closed. Consequently, the assertion follows from the cited result. Conversely, let a global minimizer (x^*, u^*, w^*) of $(\mathrm{Q})_2$ be given. Then the cited theorem ensures that (x^*, u^*, w^*) is feasible in $(\mathrm{Q})_1$ and forms even a global minimizer there. ∎

b) Necessary conditions in form of Pontryagin's principle.

Theorem 5.4. (Pontryagin's principle for $(\mathrm{P})_0$, polyconvex restriction) *Consider the problem* $(\mathrm{P})_0$ *under Assumptions 3.1., 1), 2) and 5) and Assumptions 4.10. and choose for the polyconvex set* P *a compact, convex representative* $\mathrm{Q} \subset \mathbb{R}^{nm} \times \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)}$. *Further, choose for the integrand* $f(s, \xi, v)$ *in* $(\mathrm{P})_0$ *a convex representative* $g(s, \xi, v, \omega)$ *in accordance with the assumptions mentioned above. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then there exist multipliers* $\lambda_0 \geqslant 0$, $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^{nm})$, $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R}^{\sigma(2)})$, $y^{(3)} \in L^{p/(p-3)}(\Omega, \mathbb{R}^{\sigma(3)})$, ... , $y^{(n\wedge m)} \in L^{p/(p-(n\wedge m))}(\Omega,\, \mathbb{R}^{\sigma(n\wedge m)})$ *such that the following conditions are satisfied:*

$$(\mathcal{M})\quad \lambda_0 \int_\Omega \Big(g(\,s, x^*(s), u(s), w(s)\,) - g(\,s, x^*(s), u^*(s), w^*(s)\,) \Big)\, ds - \int_\Omega \big(u(s) - u^*(s) \big)^{\mathrm{T}} y^{(1)}(s)\, ds \tag{5.21}$$

$$+ \sum_{r=2}^{(n\wedge m)} \int_\Omega \big(w_r(s) - w_r^*(s) \big)^{\mathrm{T}} y^{(r)}(s)\, ds - \sum_{r=2}^{(n\wedge m)} \int_\Omega \nabla_v \operatorname{adj}_r(\, u^*(s)\,)\, \big(u(s) - u^*(s) \big)^{\mathrm{T}} y^{(r)}(s)\, ds \geqslant 0$$

$$\forall\, (u, w) \in \Big(\mathrm{U} \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)}) \Big) \cap \mathrm{W}\,;$$

$$(\mathcal{K})\quad \lambda_0 \sum_{i=1}^n \int_\Omega \frac{\partial g}{\partial \xi_i}(\,s, x^*(s), u^*(s), w^*(s)\,)\, \big(x_i(s) - x_i^*(s) \big)\, ds \tag{5.22}$$

$$+ \sum_{i=1}^n \sum_{j=1}^m \int_\Omega \Big(\frac{\partial x_i}{\partial s_j}(s) - \frac{\partial x_i^*}{\partial s_j}(s) \Big)\, y_{ij}^{(1)}(s)\, ds = 0 \quad \forall\, x \in {W_0^{1,p}}(\Omega, \mathbb{R}^n)\,.$$

The function sets U *and* W *are defined by means of* K *and* Q *through* (3.12) *and* (3.23).

Let us define the set

$$\mathrm{Q}' = \big\{ \big(\omega_2, \omega_3, \ldots, \omega_{(n\wedge m)} \big) \in \mathbb{R}^{\sigma(2)} \times \mathbb{R}^{\sigma(3)} \times \ldots \times \mathbb{R}^{\sigma(n\wedge m)} \;\;\big|\; \big(v, \omega_2, \omega_3, \ldots, \omega_{(n\wedge m)} \big) \in \mathrm{Q} \big\}\,. \tag{5.23}$$

Proposition 5.5. (Occurence of the regular case) *Consider the problem* $(\mathrm{P})_0$ *under the assumptions of Theorem 5.4. and let* (x^*, u^*) *be a global minimizer of* $(\mathrm{P})_0$. *If there exists a number* $\gamma > 0$ *such that* $\big(T_2\big(u^*(s)\big), T_3\big(u^*(s)\big), \ldots, T_{(n\wedge m)}\big(u^*(s)\big) \big) + \mathrm{K}(\mathfrak{o}, \gamma) \in \operatorname{int}(\mathrm{Q}')$ *for almost all* $s \in \Omega$ *then in the necessary optimality conditions* $(\mathcal{M})$ *and* $(\mathcal{K})$ *from Theorem 5.4. the regular case occurs, i. e.* $\lambda_0 > 0$.

The maximum condition $(\mathcal{M})$ from Theorem 5.4. implies the following condition $(\mathcal{MP})$, which holds a. e. pointwise:

Theorem 5.6. (Pointwise maximum condition for $(\mathrm{P})_0$) *Consider the problem* $(\mathrm{P})_0$ *under the assumptions of Theorem 5.4. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_0$ *then the maximum condition* $(\mathcal{M})$ *from*

Theorem 5.4. implies the following pointwise maximum condition:

$$(\mathcal{MP})\quad \lambda_0\Big(g(\,s,x^*(s),v,\omega\,)-g(\,s,x^*(s),u^*(s),w^*(s)\,)\Big)\;-\;\big(\,v-u^*(s)\,\big)^{\mathrm{T}}\,y^{(1)}(s) \tag{5.24}$$
$$+\sum_{r=2}^{(n\wedge m)}\big(\,\omega_r-w_r^*(s)\,\big)^{\mathrm{T}}\,y^{(r)}(s)\;-\;\sum_{r=2}^{(n\wedge m)}\nabla_v\,\mathrm{adj}_r(\,u^*(s)\,)\,\big(\,v-u^*(s)\,\big)^{\mathrm{T}}\,y^{(r)}(s)\;\geqslant\;0$$
$$(\forall)\,s\in\Omega\quad\forall\,\big(\,v,\omega_2,\,\omega_3,\,\dots\,,\,\omega_{(n\wedge m)}\,\big)\in\big(\,\mathrm{K}\times\mathbb{R}^{\sigma(2)}\times\mathbb{R}^{\sigma(3)}\times\dots\times\mathbb{R}^{\sigma(n\wedge m)}\,\big)\cap\mathrm{Q}\,.$$

c) Proof of Theorems 5.4. – 5.6.

Proof of Theorem 5.4. The proof of Theorem 5.4. runs parallel to the proof of Theorem 4.3. above, which is subjected to the following modifications. In *Step 1*, within the definition of the set C, the restriction $(u,w)\in\mathrm{W}$ must be added. Then Proposition 4.5. remains true since W is convex together with K and the convex representative Q of P. *Steps 2 – 4.4.* can be carried over without alterations. In *Step 4.5.*, we must compute instead

$$\lim_{\lambda\to 0+0}\frac{1}{\lambda}\Big(\widetilde{G}\big(\,x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,, \tag{5.25}$$
$$w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\,\big)\;-\;\widetilde{G}(x^*,u^*,w^*)\Big)$$
$$=\lim_{\lambda\to 0+0}\frac{1}{\lambda}\Big(G\big(\,x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,, \tag{5.26}$$
$$w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\,\big)\;-\;G(x^*,u^*,w^*)\Big)$$
$$+\lim_{\lambda\to 0+0}\frac{K_2}{\lambda}\Big(\mathrm{Dist}\,\Big(\,\big(\,x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\,\big)\,,$$
$$L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\Big)\;-\;\mathrm{Dist}\,\Big(\,\big(\,x^*,\,u^*,\,w^*\,\big)\,,\,L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\,\Big)\Big)\;\geqslant\;0\,.$$

Since (x^*,u^*,w^*) and $(\tilde{x},\tilde{u},\tilde{w})$ belong to the convex set $L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}$, we have

$$\mathrm{Dist}\,\Big(\,\big(\,x^*,\,u^*,\,w^*\,\big)\,,\,L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\,\Big)\;=\;0\,; \tag{5.27}$$
$$\mathrm{Dist}\,\Big(\,\big(\,x^*+\lambda\,(\tilde{x}-x^*)\,,\,u^*+\lambda\,(\tilde{u}-u^*)\,,\,w^*+\lambda\,(\tilde{w}-w^*)\,\big)\,,\,L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\,\Big)\;=\;0\,, \tag{5.28}$$

and in the last term of (5.26), the expression (5.27) may be replaced by (5.28). Thus we get (5.29)

$$\dots+\lim_{\lambda\to 0+0}\frac{K_2}{\lambda}\Big(\mathrm{Dist}\,\Big(\,\big(\,x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\,\big)\,,$$
$$L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\Big)-\mathrm{Dist}\,\Big(\,\big(\,x^*+\lambda\,(\tilde{x}-x^*)\,,\,u^*+\lambda\,(\tilde{u}-u^*)\,,\,w^*+\lambda\,(\tilde{w}-w^*)\,\big)\,,\,L^p(\Omega,\mathbb{R}^n)\times\mathrm{W}\,\Big)\Big)\;\geqslant\;0$$
$$\Longrightarrow\;\lim_{\lambda\to 0+0}\frac{1}{\lambda}\Big(G\big(\,x^*+\lambda\,(x^0-x^*)\,,\,u^*+\lambda\,(u^0-u^*)\,,\,w^*+\lambda\,(w^0-w^*)+R(w^0,\lambda)-S(u^*,u^0,\lambda)\,\big) \tag{5.30}$$
$$-\,G(x^*,u^*,w^*)\Big)\;\geqslant\;-\lim_{\lambda\to 0+0}\frac{K_2}{\lambda}\cdot\big|\,\mathrm{Dist}\,(\,\dots\,)-\mathrm{Dist}\,(\,\dots\,)\,\big|$$
$$\geqslant\;-\lim_{\lambda\to 0+0}\frac{K_2}{\lambda}\Big(\lambda\,\|\,\tilde{x}-x^0\,\|_{L^p}+\lambda\,\|\,\tilde{u}-u^0\,\|_{L^p} \tag{5.31}$$
$$+\sum_{r=2}^{(n\wedge m)}\Big(\lambda\,\|\,\tilde{w}_r-w_r^0\,\|_{L^{p/r}}+\|\,R_r(w^0,\lambda)\,\|_{L^{p/r}}+\|\,S_r(u^*,u^0,\lambda)\,\|_{L^{p/r}}\,\Big)\Big)$$

since the distance function to a closed set of a normed space satisfies a Lipschitz condition with constant 1.[35] Consequently, expanding the left-hand side of (5.30) as in (4.75) – (4.77) and using (4.41), (4.45), (4.47) and (4.49), we find that the first component ϱ of the element from Step 4.1. satisfies

$$\varrho = \tilde{\varepsilon} + D_xG(x^*, u^*, w^*)\,(\tilde{x} - x^*) + D_uG(x^*, u^*, w^*)\,(\tilde{u} - u^*) + D_wG(x^*, u^*, w^*)\,(\tilde{w} - w^*) \tag{5.32}$$

$$\geqslant -\Big(\big\|\, D_xG(x^*, u^*, w^*)\,\big\| + K_2\Big)\cdot \|\, x^0 - \tilde{x}\,\|_{L^p} - \Big(\big\|\, D_uG(x^*, u^*, w^*)\,\big\| + K_2\Big)\cdot \|\, u^0 - \tilde{u}\,\|_{L^p} \tag{5.33}$$

$$- \Big(\big\|\, D_wG(x^*, u^*, w^*)\,\big\| + K_2\Big)\cdot \sum_{r=2}^{(n\wedge m)} \|\, w_r^0 - \tilde{w}_r\,\|_{L^{p/r}}$$

$$- \lim_{\lambda\to 0+0} K_2 \sum_{r=2}^{(n\wedge m)} \lambda^{-1}\Big(\|\, R_r(w^0, \lambda)\,\|_{L^{p/r}} + \|\, S_r(u^*, u^0, \lambda)\,\|_{L^{p/r}}\Big)$$

$$\geqslant -C_0\,\eta\,\Big(2\,C_1 + 1 + \sum_{r=2}^{(n\wedge m)} (1 + C_r)\Big) \geqslant -K_0\,\eta \tag{5.34}$$

with a sufficiently large number $K_0 > 0$, and Proposition 4.7. holds still true. Now *Steps 5 and 6* remain unchanged while *Step 7* must be dropped. ■

Proof of Theorem 5.5. Let us assume, on the contrary, that $\lambda_0 = 0$. Then, inserting $u = u^*$ into the maximum condition (5.21), we obtain the inequality

$$\sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, w_r - w_r^* \,\rangle = \sum_{r=2}^{(n\wedge m)} \langle\, y^{(r)},\, w_r - T_r(\,u^*\,)\,\rangle \geqslant 0\,, \tag{5.35}$$

which holds true for all functions w belonging to elements $(u, w) \in \mathrm{W} \cap \big(\,\mathrm{U} \times L^{p/2}(\Omega, \mathbb{R}^{\sigma(2)}) \times L^{p/3}(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^{p/(n\wedge m)}(\Omega, \mathbb{R}^{\sigma(n\wedge m)})\big)$. By assumption, we are allowed to insert into (5.35) arbitrary functions $w \in L^\infty(\Omega, \mathbb{R}^{\sigma(2)}) \times L^\infty(\Omega, \mathbb{R}^{\sigma(3)}) \times \ldots \times L^\infty(\Omega, \mathbb{R}^{\sigma(n\wedge m)})$ with

$$\big\|\, w_r - T_r(\,u^*\,)\,\big\|_{L^\infty(\Omega, \mathbb{R}^{\sigma(r)})} \leqslant \gamma\,,\ 2 \leqslant r \leqslant (n\wedge m)\,. \tag{5.36}$$

Consequently, for $2 \leqslant r \leqslant (n\wedge m)$, $y^{(r)}$ vanishes on all functions $z \in C_0^\infty(\Omega, \mathbb{R}^{\sigma(r)}) \cap L^p(\Omega, \mathbb{R}^{\sigma(r)})$ and thus on the whole space $L^p(\Omega, \mathbb{R}^{\sigma(r)})$, cf. [Adams/Fournier 07], p. 38, Corollary 2.30., and we get $y^{(2)}$, $y^{(3)}$, ... , $y^{(n\wedge m)} = \mathfrak{o}$. Now the argumentation may be completed as in the proof of Theorem 4.3., Step 7. ■

Proof of Theorem 5.6. This proof runs in complete analogy to the proof of Theorem 4.4. ■

d) Remarks and generalizations.

Our first remark concerns the polyconvex set P. Here Assumption 3.1., 2) may be weakened as follows:

Corollary 5.7. (General polyconvex restriction set) *Lemma 5.1. and Proposition 5.3. as well as Theorem 5.4., Proposition 5.5. and Theorem 5.6. remain true as far as the polyconvex set* $\mathrm{P} \subseteq \mathbb{R}^{nm}$ *in Assumption 3.1., 2) is closed but possibly unbounded.*

Proof. Since $\mathrm{K} \subset \mathbb{R}^{nm}$ is convex and compact, we may replace P by $\widetilde{\mathrm{P}} = \mathrm{K} \cap \mathrm{P}$ before starting the analysis of the problems. This causes no change in the feasible domains. However, the set $\widetilde{\mathrm{P}}$ is compact together with K and polyconvex as an intersection of a convex and a polyconvex set. ■

Remark 5.8. As a consequence of the consideration of a polyconvex gradient constraint, the number of variables in $(\mathrm{Q})_1$ and $(\mathrm{Q})_2$ as well as in the conditions of Pontryagin's principle cannot be reduced even if the

[35] [Clarke 90], p. 50, Proposition 2.4.1.

integrand does not depend explicitly on some of them. For the same reason, in contrast to (4.9) – (4.12), the pointwise condition ($\mathcal{MP}$) from Theorem 5.6. allows for a decomposition into separate conditions in special cases only.

Remark 5.9. Theorems 5.4. and 5.6. may be restated for strong local minimizers (x^*, u^*) of $(\mathrm{P})_0$, cf. [Ioffe/ Tichomirow 79] , p. 98 f.

Remark 5.10. For the purposes of optimization, it would be desirable to know the *largest* possible convex representative of a given polyconvex set P, thus obtaining maximal significance of the conditions ($\mathcal{M}$) and ($\mathcal{MP}$).

6. Application to hyperelastic image registration.

a) Unimodal image registration.

Assume that on a two- or three-dimensional domain $\Omega \subset \mathbb{R}^m$, $m \in \{2, 3\}$, two greyscale images are given, which will be identified with at least measurable functions I_0, $I_1 : \Omega \to [0, 1]$. Considering I_0 as reference image, one searches for a deformation field $x : \Omega \to \mathbb{R}^m$ satisfying the condition $I_1(s - x(s)) \approx I_0(s)$, thus modifying the template image I_1 such that it matches the reference image I_0 in a best possible way. In this abstract formulation of the registration problem, the single assumption is required that there is an overall correlation between the greyscale intensity distributions as well as the geometrical properties of the template and reference image. For the practical determination of a possible deformation field x as well as for a reliable interpretation of the result, more information about the pictured objects and their motion behaviour is needed. [36]

In numerous situations, a reasonable approach to unimodal registration is to attribute the changes in I_1 with respect to the reference image I_0 to an *elastic deformation* of the pictured objects. This is particularly true for the imaging of living tissue, which behaves according to hyperelastic material laws. [37] Consequently, a large part of the literature is concerned with variational or PDE methods where x is sought as a linear-elastic [38] or hyperelastic deformation. [39] In the latter case, the problems involve polyconvex stored-energy functions. [40]

The interest in an optimal control access to the elastic registration problem is caused by the fact that the validity of the underlying elasticity models depends crucially on the uniform boundedness of the maximal shear stress generated by the deformation x. [41] Consequently, it is advisable to incorporate restrictions for the partial derivatives of x into the given variational models. In the present paper, we confine ourselves to convex restriction sets. However, in a forthcoming publication we will extend our approach to polyconvex control restrictions as $0 < R_0 \leqslant \det(J(x)) \leqslant R_1 < \infty$. In the following, we reformulate a two-dimensional registration problem within the framework of optimal control and provide the necessary optimality conditions for the problem.

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[36] A detailed introduction to the registration problem may be found in [Hintermüller/Keeling 09] , [Modersitzki 04] and [Modersitzki 09] .

[37] See e. g. [Ogden 03] .

[38] We refer e. g. to [Fischer/Modersitzki 03] , [Haber/Modersitzki 04] , [Henn/Witsch 00] , [Henn/Witsch 01] and [Modersitzki 04] , pp. 77 ff.

[39] See e.g. [Burger/Modersitzki/Ruthotto 13] , [Droske/Rumpf 04] , [Droske/Rumpf 07] and [Le Guyader/Vese 09] .

[40] Examples may be found in [Balzani/Neff/Schröder/Holzapfel 06] .

[41] This is even true for living tissue, cf. [Gasser/Holzapfel 02] , p. 340 f., and the literature cited therein.

b) A two-dimensional registration problem with polyconvex regularizer.

Let us consider the following two-dimensional image registration problem with polyconvex regularizer: [42)]

$$(\mathrm{R})_2: \quad F(x) = \int_\Omega \Big(I_1(\, s - x(s)\,) - I_0(s) \Big)^2 \, ds \, + \, \mu \cdot \int_\Omega \Big(c_1 \, \| \, Jx(s) \, \|^{\,p} \tag{6.1}$$

$$+ c_2 \left(\det \left(\, Id_2 - Jx(s) \, \right) - 1 \, \right)^2 \Big) \, ds \, \longrightarrow \, \inf! \, ;$$

$$x \in W_0^{1,p}(\Omega, \mathbb{R}^2) \, ; \; Jx(s) \in \mathrm{K} = \mathrm{P} \subset \mathbb{R}^{2\times 2} \;\; (\forall) \, s \in \Omega \, . \tag{6.2}$$

As discussed in [Burger/Modersitzki/Ruthotto 13] and [Droske/Rumpf 04], the objective can be regarded as a stored-energy functional, which is connected with a generic hyperelastic model. We assume that $4 \leqslant p < \infty$, μ, c_1, $c_2 > 0$. The image data I_0 and I_1 belong to $L^\infty(\Omega, \mathbb{R})$ and $C_0^1(\Omega, \mathbb{R})$, respectively. $\mathrm{K} = \mathrm{P} \subset \mathbb{R}^{2\times 2}$ is a compact convex set with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. We use the matrix norm $\| \left(\begin{smallmatrix} v_1 & v_2 \\ v_3 & v_4 \end{smallmatrix}\right) \|^p = |\, v_1 \,|^p + |\, v_2 \,|^p + |\, v_3 \,|^p + |\, v_4 \,|^p$, which is continuously differentiable with respect to its arguments since $p \geqslant 4$. Id_2 denotes the $(2,2)$-unit matrix.

In [Wagner 10] and [Wagner 12], after an appropriate approximation of the data term, a direct method was employed for the numerical solution of this problem. In [Angelov 11] and [Angelov/Wagner 12], p. 5 f., the problem $(\mathrm{R})_2$ was incorporated into a larger scheme for hyperelastic registration of multimodal image data. Applying Theorems 4.1. and 4.2. to $(\mathrm{R})_2$, we obtain the following set of necessary optimality conditions:

Proposition 6.1. (Pontryagin's principle for $(\mathrm{R})_2$ **)** *Consider* $(\mathrm{R})_2$ *under the analytical assumptions mentioned above. If* (x^*, u^*) *is a global minimizer of* $(\mathrm{R})_2$ *then there exist multipliers* $\lambda_0 > 0$*,* $y^{(1)} \in L^{p/(p-1)}(\Omega, \mathbb{R}^4)$ *and* $y^{(2)} \in L^{p/(p-2)}(\Omega, \mathbb{R})$ *such that the following conditions are satisfied:*

$$(\mathcal{MP})_1 \quad \lambda_0 \, \mu \, c_1 \sum_{i=1}^2 \sum_{j=1}^2 \Big(\, |\, v_{ij} \,|^p - |\, u^*_{ij}(s) \,|^p \Big) + \lambda_0 \, \mu \, c_2 \Big(\left(\, v_{11} + v_{22} \, \right)^2 - \left(\, u^*_{11}(s) + u^*_{22}(s) \, \right)^2 \tag{6.3}$$

$$+ \; 2 \, \det u^*(s) \left(\, v_{11} + v_{22} - u^*_{11}(s) - u^*_{22}(s) \, \right) \Big)$$

$$- \sum_{i=1}^2 \sum_{j=1}^2 \Big(\, y^{(1)}_{ij}(s) + \frac{\partial}{\partial v_{ij}} \det \left(\, u^*(s) \, \right) y^{(2)}(s) \Big) \left(\, v_{ij} - u^*_{ij}(s) \, \right) \, \geqslant \, 0 \quad (\forall) \, s \in \Omega \quad \forall \, v \in \mathrm{K} \, ;$$

$$(\mathcal{MP})_2 \quad \lambda_0 \, \mu \, c_2 \Big(\left(\, \omega_2 \, \right)^2 - \left(\, \det u^*(s) \, \right)^2 - 2 \left(\, u^*_{11}(s) + u^*_{22}(s) \, \right) \left(\, \omega_2 - \det u^*(s) \, \right) \Big) \tag{6.4}$$

$$+ \left(\, \omega_2 - \det u^*(s) \, \right) y^{(2)}(s) \, \geqslant \, 0 \quad (\forall) \, s \in \Omega \quad \forall \, \omega_2 \in \mathbb{R} \, ;$$

$$(\mathcal{K}) \quad - \lambda_0 \int_\Omega \Big(\frac{\partial I_1}{\partial s_1}(\, s - x^*(s) \,) \left(\, x_1(s) - x^*_1(s) \, \right) + \frac{\partial I_1}{\partial s_2}(\, s - x^*(s) \,) \left(\, x_2(s) - x^*_2(s) \, \right) \Big) \, ds \tag{6.5}$$

$$+ \sum_{i=1}^2 \sum_{j=1}^2 \int_\Omega \Big(\frac{\partial x_i}{\partial s_j}(s) - \frac{\partial x^*_i}{\partial s_j}(s) \Big) \, y^{(1)}_{ij}(s) \, ds \, = \, 0 \quad \forall \, x \in W_0^{1,p}(\Omega, \mathbb{R}^2) \, .$$

Proof. In order to apply Theorems 4.11., 4.1. and 4.2. to $(\mathrm{R})_2$, we must to verify that the data of the problem satisfy assumptions 3)$'$ and 4)$'$ from Remark 3.4. as well as the growth conditions (4.96) – (4.98). Obviously, assumption 3)$'$ from Remark 3.4. is satisfied. For the polyconvex integrand $f(s, \xi, v)$, we choose the convex representative $g\colon\; \Omega \times \mathbb{R}^2 \times \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ defined as

$$g(s, \xi, v, \omega_2) \; = \; \left(\, I_1(s - \xi) - I_0(s) \, \right)^2 + \mu \, c_1 \sum_{i=1}^2 \sum_{j=1}^2 |\, v_{ij} \,|^p + \mu \, c_2 \left(\, \omega_2 - v_{11} - v_{22} \, \right)^2 \tag{6.6}$$

42) Sligthly modified from [Wagner 11], p. 218, (4.15), and [Wagner 10], p. 5, (2.16) – (2.19). Note that the reference points within the regularization term must be chosen in accordance with the deviation of $s - x(s)$ from the identity.

with the partial derivatives

$$\frac{\partial g}{\partial \xi_i}(s,\xi,v,\omega_2) \;=\; -2\left(I_1(s-\xi) - I_0(s) \right) \frac{\partial I_1}{\partial s_i}(s-\xi)\,,\;\; 1 \leqslant i \leqslant 2\,; \tag{6.7}$$

$$\frac{\partial g}{\partial v_{11}}(s,\xi,v,\omega_2) \;=\; p\,\mu\,c_1 \left| \, v_{11} \, \right|^{p-1} - 2\,\mu\,c_2 \left(\omega_2 - v_{11} - v_{22} \right); \tag{6.8}$$

$$\frac{\partial g}{\partial v_{12}}(s,\xi,v,\omega_2) \;=\; p\,\mu\,c_1 \left| \, v_{12} \, \right|^{p-1}; \tag{6.9}$$

$$\frac{\partial g}{\partial v_{21}}(s,\xi,v,\omega_2) \;=\; p\,\mu\,c_1 \left| \, v_{21} \, \right|^{p-1}; \tag{6.10}$$

$$\frac{\partial g}{\partial v_{22}}(s,\xi,v,\omega_2) \;=\; p\,\mu\,c_1 \left| \, v_{22} \, \right|^{p-1} - 2\,\mu\,c_2 \left(\omega_2 - v_{11} - v_{22} \right); \tag{6.11}$$

$$\frac{\partial g}{\partial \omega_2}(s,\xi,v,\omega_2) \;=\; 2\,\mu\,c_2 \left(\omega_2 - v_{11} - v_{22} \right). \tag{6.12}$$

Let us confirm the growth condition (3.5), thus establishing assumption $4)'$ from Remark 3.4. Since $I_0(s)$ is essentially bounded on Ω and $I_1(s)$, after extension by zero to $\mathbb{R}^2 \setminus \Omega$, is bounded on $\mathbb{R}^2$, we get for almost all $s \in \Omega$ and for all $(\xi, v, \omega_2) \in \mathbb{R}^2 \times \mathbb{R}^4 \times \mathbb{R}$ (6.13)

$$\begin{aligned} \left| \, g(s,\xi,v,\omega_2) \, \right| \;&\leqslant\; C_1 \left(\left| \, I_0(s) \, \right|^2 + \left| \, I_1(s-\xi) \, \right|^2 \right) + \,\mu\,c_1 \sum_{i=1}^{2} \sum_{j=1}^{2} \left| \, v_{ij} \, \right|^p + \mu\,c_2\,C_2 \left(\left| \, v_{11} \, \right|^2 + \left| \, v_{22} \, \right|^2 + \left| \, \omega_2 \, \right|^2 \right) \\ &\leqslant\; 2\,C_3 + 4\,\mu\,c_1 \left| \, v \, \right|^p + \mu\,c_2\,C_2 \left(2 \left| \, v \, \right|^2 + \left| \, \omega_2 \, \right|^2 \right) \;\leqslant\; 2\,C_3 + C_4 \left(1 + \left| \, v \, \right|^p + \left| \, \omega_2 \, \right|^{p/2} \right) \end{aligned} \tag{6.14}$$

since $p \geqslant 4$. Thus (3.5) is satisfied with $A_0(s) \equiv C_3$ and $B_0(\xi) \equiv C_3$. Further, we have

$$\left| \, \frac{\partial g}{\partial \xi_i}(s,\xi,v,\omega_2) \, \right| \;\leqslant\; 2 \left(\left| \, I_1(s-\xi) \, \right| + \left| \, I_0(s) \, \right| \right) \| \, I_1 \, \|_{C^1} \;\leqslant\; 2\,C_5 \, \| \, I_1 \, \|_{C^1}\,,\;\; 1 \leqslant i \leqslant 2\,, \tag{6.15}$$

and (4.96) is satisfied with $A_i(s) \equiv C_5 \, \| \, I_1 \, \|_{C^1}$ and $B_i(\xi, v, \omega_2) \equiv C_5 \, \| \, I_1 \, \|_{C^1}$, $1 \leqslant i \leqslant 2$. Concerning (4.97) and (4.98), we see that the right-hand sides in the inequalities

$$\left| \, \frac{\partial g}{\partial v_{11}}(s,\xi,v,\omega_2) \, \right| \;\leqslant\; p\,\mu\,c_1 \left| \, v_{11} \, \right|^{p-1} + 2\,\mu\,c_2 \left(2 \left| \, v \, \right| + \left| \, \omega_2 \, \right| \right), \tag{6.16}$$

$$\left| \, \frac{\partial g}{\partial v_{12}}(s,\xi,v,\omega_2) \, \right| \;=\; p\,\mu\,c_1 \left| \, v_{12} \, \right|^{p-1}, \tag{6.17}$$

$$\left| \, \frac{\partial g}{\partial \omega_2}(s,\xi,v,\omega_2) \, \right| \;\leqslant\; 2\,\mu\,c_2 \left(2 \left| \, v \, \right| + \left| \, \omega_2 \, \right| \right) \tag{6.18}$$

are measurable and bounded on bounded subsets of $\mathbb{R}^2 \times \mathbb{R}^4 \times \mathbb{R}$. The derivatives $\partial g(s,\xi,v,\omega_2)/\partial v_{21}$ and $\partial g(s,\xi,v,\omega_2)/\partial v_{22}$ can be estimated in analogous way. Consequently, (4.97) and (4.98) hold with $A_l^{(1)}(s) \equiv 0$, $A^{(2)}(s) \equiv 0$ and $B_l^{(1)}(\xi, v, \omega_2)$, $1 \leqslant l \leqslant 4$ and $B^{(2)}(\xi, v, \omega_2)$ as given through (6.16) – (6.18). Consequently, for a given global minimizer (x^*, u^*) of $(\mathrm{R})_2$, the necessary optimality conditions take the claimed form. ∎

A numerical implementation of the necessary conditions obtained here as well as further applications to problems with higher-dimensional data sets will be reserved for future publications.

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