Jensen’s inequality for the lower semicontinuous quasiconvex envelope and relaxation of multidimensional control problems

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1. Introduction.

a) Relaxation of multidimensional control problems.

Relaxation of a given variational or control problem means to define a new problem, which conserves the minimal value of the original problem, whose admissible domain contains the original one (eventually in the sense of an embedding), and whose cost functional is lower semicontinuous with respect to a suitable topology. Therefore, the relaxed problem admits global minimizers, what makes possible the application of direct methods. For the relaxation of the basic problem of multidimensional calculus of variations, two different approaches may be pursued. In the first one, the integrand \( r(t, \xi, v) \) within the cost functional has to be replaced by its convex (\( n = 1 \)) resp. quasiconvex (\( n \geq 2 \)) envelope with respect to \( v \). The second approach requires the introduction of generalized controls ("Young measures") \( \mu : \Omega \rightarrow \text{rc} \text{a}^{pr} (\mathbb{R}^{nm}) \).

Then the objective in (1.1) must be replaced by

\[
\tilde{F}(x, \mu) = \int_\Omega \int_{\mathbb{R}^{nm}} r(t, x(t), v) d\mu(t) \quad \text{with} \quad \frac{\partial x_i}{\partial t_j} = \int_{\mathbb{R}^{nm}} v_{ij} d\mu(t) \quad \forall i, j \quad (\forall) t \in \Omega.
\]

The connecting link between the both relaxation approaches is the Jensen’s integral inequality for the convex resp. quasiconvex envelope.

In the present paper, we apply these ideas to multidimensional control problems of Dieudonné-Rashevsky type. Problems of this kind will be obtained by addition of restrictions for the partial derivatives of \( x \) to the variational problem (V). They find applications e.g. in models for the torsion of prismatic bars, in optimization problems for convex bodies under geometrical restrictions and within the framework of image processing. In their papers on underdetermined boundary value problems for nonlinear first-order PDE’s, Dacorogna and Marcellini arrived at Dieudonné-Rashevsky type problems as well.

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01) Cf. [Buttazzo 89], pp. 2 ff. and pp. 16 ff., [Roubiček 97], pp. vii ff.
02) Cf. [Morrey 66], pp. 15 ff., and [Dacorogna 89], pp. 4 ff.
03) [Dacorogna 89], pp. 228 ff., Theorem 2.1., and pp. 235 ff., Corollaries 2.2. and 2.3.
04) Cf. [Wagner 07d], p. 7, Definition 2.5.
05) [Pedregal 97], pp. 65 ff., particularly Theorem 4.4.
07) [Lur’e 75], pp. 240 ff., [Ting 69a], p. 531 ff., [Ting 69b], [Wagner 96], pp. 76 ff.
08) [Andrejewa/Klötzler 84a], [Andrejewa/Klötzler 84b], p. 149 f.
09) [Brune 07], [Franek 07a], [Franek 07b], [Wagner 07b].
10) [Dacorogna/Marcellini 97], [Dacorogna/Marcellini 98], [Dacorogna/Marcellini 99].
It is a well-known fact from the relaxation theory of multidimensional variational problems that the investigation of general integrands \( r(t, \xi, v) \) can be reduced to the special case where the integrand depends on \( v \) only.\(^{11}\) For this reason, in the present paper we confine ourselves to the investigation of a model problem with an integrand \( f(v) \), which may be formulated as follows:

\[
(P) \quad F(x) = \int_{\Omega} f(Jx(t)) \, dt \quad \longrightarrow \quad \inf !; \quad x \in W^{1, \infty}_0(\Omega, \mathbb{R}^n); \quad (1.3)
\]

\[
Jx(t) = \begin{pmatrix}
\frac{\partial x_1}{\partial t_1}(t) & \ldots & \frac{\partial x_1}{\partial t_m}(t) \\
\vdots & & \vdots \\
\frac{\partial x_n}{\partial t_1}(t) & \ldots & \frac{\partial x_n}{\partial t_m}(t)
\end{pmatrix} \in K \subset \mathbb{R}^{n \times m} \quad (\forall) \, t \in \Omega. \quad (1.4)
\]

Here we choose \( n \geq 1 \) (and particularly \( n > 1 \)), \( m \geq 2 \), \( \Omega \subset \mathbb{R}^m \) as the closure of a bounded strongly Lipschitz domain with \( \sigma \in \text{int}(\Omega) \), a convex body \( K \subset \mathbb{R}^{nm} \) with \( \sigma \in \text{int}(K) \) and an integrand \( f: \mathbb{R}^{nm} \to \mathbb{R} = \mathbb{R} \cup \{+\infty\} \) with \( f \big|_{(\mathbb{R}^{nm} \setminus K)} \equiv +\infty \), whose restriction to \( K \) is continuous.

b) The lower semicontinuous quasiconvex envelope.

In order to extend the known relaxation results in multidimensional control\(^ {12}\) to the vectorial case \((n \geq 2)\), the author introduced an appropriate quasiconvex envelope for unbounded integrands \( f: \mathbb{R}^{nm} \to \mathbb{R} \). For such functions, the notion of quasiconvexity has to be precised in the following way:

**Definition 1.1. (Quasiconvex function with values in \( \mathbb{R} )\)**\(^ {13}\) A function \( f: \mathbb{R}^{nm} \to \mathbb{R} \) with the following properties is said to be quasiconvex:

1) \( \text{dom} (f) \subseteq \mathbb{R}^{nm} \) is a (nonempty) Borel set;

2) \( f \big|_{\text{dom} (f)} \) is Borel measurable and bounded from below on every bounded subset of \( \text{dom} (f) \);

3) for all \( v \in \mathbb{R}^{nm} \), \( f \) satisfies Morrey’s integral inequality:

\[
f(v) \leq \frac{1}{|\Omega|} \int_{\Omega} f(v + Jx(t)) \, dt \quad \forall x \in W^{1, \infty}_0(\Omega, \mathbb{R}^n); \quad (1.5)
\]

or equivalently

\[
f(v) = \inf \left\{ \frac{1}{|\Omega|} \int_{\Omega} f(v + Jx(t)) \, dt \mid x \in W^{1, \infty}_0(\Omega, \mathbb{R}^n) \right\}. \quad (1.6)
\]

Here \( \Omega \subset \mathbb{R}^m \) is the closure of a bounded strongly Lipschitz domain.

Then the lower semicontinuous quasiconvex envelope of an unbounded function may be defined as follows:

\(^{11}\) [Dacorogna 89], pp. 157 ff. and 228 ff.

\(^{12}\) The most important among these results is Ekeland/Témam’s relaxation theorem, see [Ekeland/Témam 99], p. 327, Corollary 2.17., together with p. 334, Proposition 3.4., and p. 335 f., Proposition 3.6. Extensions of this theorem with control restrictions of the shape \( u \in U = \{ u \in L^p(\Omega, \mathbb{R}^{nm}) \mid u(t) \in K(t) \quad (\forall) \, t \in \Omega \} \) have been proved by De Arcangelis et al. but do not exceed the case \( n = 1 \) as well. See e. g. [De Arcangelis/Monsurrò/Zappale 04], p. 386, Theorem 6.6., and [De Arcangelis/Zappale 05]., pp. 267 ff. Section 5.

\(^{13}\) [Wagner 07], p. 6, Definition 2.9., as specification of [Ball/Murat 84], p. 228, Definition 2.1., in the case \( p = (+\infty) \).
Definition 1.2. (Lower semicontinuous quasiconvex envelope $f^{qc}$ for functions with values in $\mathbb{R}$) \(^{14}\) For any function $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ bounded from below, we define

$$f^{qc}(v) = \sup \{ g(v) \mid g : \mathbb{R}^{nm} \rightarrow \mathbb{R} \text{ quasiconvex and lower semicontinuous}, \quad g(v) \leq f(v) \ \forall v \in \mathbb{R}^{nm} \}. \quad (1.7)$$

Obviously, Definition 1.2. generalizes the usual formation of a quasiconvex envelope since any quasiconvex function $g$ below a finite function $f$ is continuous from the outset. \(^{15}\)

c) Two representation theorems for $f^{qc}$.

In the situation appearing in (P), the author proved two representation theorems for $f^{qc}$. The first one describes the envelope in terms of Jacobi matrices:

Theorem 1.3. (First representation theorem for $f^{qc}$) \(^{16}\) Let a convex body $K \subset \mathbb{R}^{nm}$ with $\sigma \in \text{int}(K)$ and a function $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ with $f \big| K \in C^0(K, \mathbb{R})$ and $f \big| (\mathbb{R}^{nm} \setminus K) \equiv (+\infty)$ be given. Then its lower semicontinuous quasiconvex envelope $f^{qc} : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ admits the representation

$$f^{qc}(v_0) = \begin{cases} f^*(v_0) & v_0 \in \text{int}(K); \\ \lim_{v \rightarrow v_0, v \in R \cap \text{int}(K)} f^*(v) & v_0 \in \partial K; \\ (+\infty) & v_0 \in \mathbb{R}^{nm} \setminus K, \end{cases} \quad (1.8)$$

where $R = \overrightarrow{0 v_0}$ denotes the ray through $v_0$ starting from the origin, and $f^*(v_0)$ is defined by

$$f^*(v_0) = \inf \left\{ \frac{1}{|\Omega|} \int_{\Omega} f(v_0 + Jx(t)) \, dt \mid x \in W^{1,\infty}_0(\Omega, \mathbb{R}^n), v_0 + Jx(t) \in K \ (\forall t \in \Omega) \right\} \in \mathbb{R}. \quad (1.9)$$

In order to formulate the second theorem, which shows how to represent $f^{qc}$ in terms of probability measures, we introduce the following set of probability measures:

Definition 1.4. (The set-valued map $S^{qc}$) \(^{17}\) For $v_0 \in K$, let us define the following set of probability measures:

$$S^{qc}(v_0) = \{ \nu \in \text{rca}^{pr}(K) \mid \text{there exist sequences } \{ v^N \}, \text{int}(K) \text{ and } \{ x^N \}, W^{1,\infty}_0(\Omega, \mathbb{R}^n) \text{ with} \quad (1.10)$$

c) $\lim_{N \rightarrow \infty} v^N = v_0$.

b) $\lim_{N \rightarrow \infty} \| x^N \|_{C^0(\Omega, \mathbb{R}^n)} = 0$.

c) $v^N + Jx^N(t) \in K \ (\forall t \in \Omega) \forall N \in \mathbb{N}$.

d) $\{ v^N + Jx^N \}$ generates the constant generalized gradient control $\nu = \{ \nu \}$.

Theorem 1.5. (Second representation theorem for $f^{qc}$) \(^{18}\) Take over the assumptions of Theorem 1.3. Then with the set-valued map $S^{qc} : K \rightarrow \mathbb{R}\{ \text{rca}^{pr}(K) \}$ from Definition 1.4., it holds for all $v_0 \in K$ that

$$f^{qc}(v_0) = \min \left\{ \int_{\mathbb{R}} f(v) \, d\nu(v) \mid \nu \in S^{qc}(v_0) \right\}. \quad (1.11)$$

\(^{14}\) [WAGNER 07A] , p. 9, Definition 2.14., 2).

\(^{15}\) [DACOROGNA 89] , p. 29, Theorem 2.3., 2).

\(^{16}\) [WAGNER 07A] , p. 29, Theorem 4.1.

\(^{17}\) Synopsis of [WAGNER 07D] , p. 15, Definition 3.1. and Lemma 3.2., and p. 21, Theorem 3.9., 2).

\(^{18}\) [WAGNER 07D] , p. 3, Theorem 1.4.
d) Main results.

Based on the second representation theorem, we will establish the *Jensen’s integral inequality for* $f^{(qc)}$. Our first main result stands in complete analogy to the Jensen’s integral inequalities for convex\(^\text{19}\) and rank one convex functions\(^\text{20}\) and generalizes a previous result of KINDERLEHRER/PEDREGAL: \(^\text{21}\)

**Theorem 1.6. (Jensens’s integral inequality for $f^{(qc)}$)** Let a convex body $K \subset \mathbb{R}^m$ with $0 \in \text{int}(K)$ and a function $f : \mathbb{R}^m \to \mathbb{R}$ with $f | K \in C^1(K, \mathbb{R})$ and $f | (\mathbb{R}^m \setminus K) \equiv (+\infty)$ be given. The set-valued map $S^{(qc)} : K \to \mathcal{P}(\text{rca}^{pr}(K))$ is taken from Definition 1.4.

1) For all $w \in K$, the following implication holds: $\nu \in S^{(qc)}(w) \implies f^{(qc)}(w) \leq \int_K f^{(qc)}(v) d\nu(v)$. (1.12)

2) For all $w \in K$, it holds that

$$f^{(qc)}(w) = f^{(qc)}\left(\begin{array}{c}
\int_K v_{11} d\nu(v) \\
\vdots \\
\int_K v_{nm} d\nu(v)
\end{array}\right) \leq \int_K f^{(qc)}\left(\begin{array}{cc}
v_{11} & \cdots & v_{1m} \\
\vdots & \ddots & \vdots \\
v_{n1} & \cdots & v_{nm}
\end{array}\right) d\nu(v) \quad \forall \nu \in S^{(qc)}(w) .$$

Our second main result is a “Young measure” relaxation theorem for (P) (Theorem 4.2.), whose detailed formulation will be postponed to Section 4 below. By means of Theorem 1.6., we prove that the problem $(P)^{(qc)}$, where only the integrand within the objective has been replaced by $f^{(qc)}$, and the problem $(\tilde{P})^{(qc)}$, which arises from (P) by introduction of generalized controls, admit the same finite minimal value. Therefore, both approaches for the quasiconvex relaxation of the multidimensional control problem (P) are connected in the same way as for the basic problem in multidimensional calculus of variations.

e) Outline of the paper.

We close Section 1 with a synopsis of notions and abbreviations. In Section 2, we introduce the metric space $\text{rca}^{pr}(K)$ of the probability measures supported on $K$ and present some basic concepts from the theory of generalized controls (“Young measures”). Section 3 starts with the formulation of a further representation theorem for $f^{(qc)}$, involving continuous, restricted quasiconvex functions. The we turn to the proof of Theorem 1.6. and draw a conclusion about the support of the measure, which realizes the minimum in Theorem 1.5. In Section 4, we compare the relaxation approaches for (P) mentioned in Section 1.a) above. Concerning the first approach with the replacement of the integrand, we recall the result from [WAGNER 07c] about the problem $(P)^{(qc)}$. Pursuing the second approach, we formulate a relaxed problem $(\tilde{P})^{(qc)}$ with generalized controls and prove then the identity of the minimal values of $(P)^{(qc)}$ and $(\tilde{P})^{(qc)}$. Besides the Jensen’s integral inequality, the main tool for this proof is a theorem of SCHÄL concerning the measurability of an “optimal” selection of a set-valued map. Together with other concepts from the theory of set-valued maps (Painlevé-Kuratowski limits, semicontinuity, continuity and measurability), this theorem will be presented in the Appendix.

f) Notations and abbreviations.

Let $k \in \{0, 1, \ldots, \infty\}$ and $1 \leq p \leq \infty$. Then $C^k(\Omega, \mathbb{R}^r)$, $L^p(\Omega, \mathbb{R}^r)$ and $W^{k,p}(\Omega, \mathbb{R}^r)$ denote the spaces of $r$-dimensional vector functions whose components are $k$-times continuously differentiable, belong to $L^p(\Omega)$

\(^{19}\) IOFFE/TICHOMIROV 79, p. 310. See also WAGNER 06a, p. 131, Theorem 10.19.


\(^{21}\) KINDERLEHRER/PEDREGAL 91, p. 342, Theorem 4.1.
or to the Sobolev space of $L^p(\Omega)$-functions with weak derivatives up to $k$th order in $L^p(\Omega)$ respectively. In addition, functions within the subspaces $C^k_0(\Omega, \mathbb{R}^r) \subset C^k(\Omega, \mathbb{R}^r)$ are compactly supported while functions within the subspace $W^{1,\infty}_0(\Omega, \mathbb{R}^r) \subset W^{1,\infty}(\Omega, \mathbb{R}^r)$ admit a (Lipschitz-) continuous representative\(^{22}\) with zero boundary values. The symbol $\partial x/\partial t_j$ may denote the classical as well as the weak partial derivative of $x$ by $t_j$. Further, we denote the Jacobi matrix of $x$ by $Jx$.

The space of Radon measures (signed regular measures) acting on the $\sigma$-algebra of the Borel subsets of a compact set $K \subset \mathbb{R}^{nm}$ is denoted by $rca(K)$. Equipped with the total variation norm $\mathcal{V}\mu(K)$, it forms a Banach space.\(^{23}\) Due to the compactness of $K$, the dual space $(C^0(K, \mathbb{R}))^*$ and $rca(K)$ are isomorphical;\(^{24}\) consequently, every linear, continuous functional on $C^0(K, \mathbb{R})$ may be represented as an integral with respect to a Radon measure $\nu \in rca(K)$. The subset of the probability measures, endowed with a suitable metric, will be denoted by $rca^{pr}(K)$ (see Definition 2.1. below). $\delta_v$ denotes the Dirac measure concentrated in $v \in K$.

We denote by $\text{int}(A)$, $\partial A$, $\text{cl}(A)$, $\text{co}(A)$ and $|A|$ the interior, boundary, closure, the convex hull and the $r$-dimensional Lebesgue null set. The symbol $o$ within the subspace or to the Sobolev space of $\mathcal{L}(\Omega)$-functions with weak derivatives up to $k$th order in $\mathcal{L}(\Omega)$ respectively.

Definition 1.7. (Function class $\mathcal{F}_K$) Let $K \subset \mathbb{R}^{nm}$ be a given convex body with $\sigma \in \text{int}(K)$. We say that a function $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ belongs to the class $\mathcal{F}_K$ iff $f \mid K \in C^0(\mathbb{R}, \mathbb{R})$ and $f \mid (\mathbb{R}^{nm} \setminus K) \equiv (+\infty)$.

Consequently, any function $f \in \mathcal{F}_K$ is bounded and uniformly continuous on $K$, and the class $\mathcal{F}_K$ and the Banach space $C^0(K, \mathbb{R})$ are isomorphical and isometrical.

If $X$ is an arbitrary set then $\mathcal{P}(X)$ denotes the set of all subsets of $X$. For the definition of the Painlevé-Kuratowski limits $\liminf^K_{N \to \infty} E_N$, $\limsup^K_{N \to \infty} E_N$ and $\lim^K_{N \to \infty} E_N$ of a set sequence $\{E_N\}$, we refer to the Appendix.

We close this subsection with three nonstandard notations. $\{x^N\}$, $A$” denotes a sequence $\{x^N\}$ with members $x^N \in A$. If $A \subset \mathbb{R}^r$ then the abbreviation “$(\forall) t \in A$’” has to be read as “for almost all $t \in A$” resp. “for all $t \in A$ except a $r$-dimensional Lebesgue null set”. The symbol $\sigma$ denotes, depending on the context, the zero element resp. the zero function of the underlying space.

2. Generalized controls and the sets $S^{(\sigma)}$.

a) The metric space $rca^{pr}(K)$ and its properties.

Throughout the whole section, let $K \subset \mathbb{R}^{nm}$ be a fixed convex body with $\sigma \in \text{int}(K)$.

Definition 2.1. (Metric space $rca^{pr}(K)$)\(^{25}\) The subset of the probability measures $\nu \in rca(K)$, endowed with the metric

$$
\sigma(\nu', \nu'') = \sum_{s=1}^{\infty} \frac{1}{2^{1+s} (1 + L_s)} \left| \int_K g_s(\nu) \left( d\nu'(v) - d\nu''(v) \right) \right|,
$$

\(2.1\)

\(^{22}\) [Evans/Gariepy 92], p. 131, Theorem 5.

\(^{23}\) [Dunford/Schwartz 88], p. 161 f.

\(^{24}\) Ibid., p. 265, Theorem 3.

\(^{25}\) [Wagner 07d], p. 4 f., Definition 2.1., Lemma 2.2. and Definition 2.3.
forms the metric space \( \text{rca}^{pr}(K) \). Here \( g_s \in C^0(K, \mathbb{R}) \cap W^{1,\infty}(K, \mathbb{R}) \) are countably many functions with \( \| g_s \|_{C^0(K, \mathbb{R})} = 1 \) and Lipschitz constants \( L_s > 0 \), which form a dense subset \( \{ g_s \} \) of the unit ball of \( C^0(K, \mathbb{R}) \) with respect to its norm topology.

**Theorem 2.2. (Properties of the metric space \( \text{rca}^{pr}(K) \))** \(^{26}\)

1) \( \text{rca}^{pr}(K) \) forms a compact metric space.

2) For every sequence \( \{ \nu^N \} \), \( \text{rca}^{pr}(K) \), it holds that \( \lim_{N \to \infty} \sigma(\nu^N, \nu) = 0 \iff \nu^N \rca K \nu \). Therefore, in \( \text{rca}^{pr}(K) \), convergence with respect to the metric \( \sigma \) is equivalent to weak\(^*\)-convergence.

3) The metric space \( \text{rca}^{pr}(K) \) is separable with the countable, dense subset

\[
\{ \sum_{k=1}^K \lambda_k \delta_{v_k} \mid \sum_{k=1}^K \lambda_k = 1, \lambda_k \in [0,1] \cap \mathbb{Q}, v_k \in K \cap \mathbb{Q}^{nm}, 1 \leq k \leq K, K \in \mathbb{N} \}.
\]

\(^{27}\) Theorem 2.6. (Properties of the spaces \( \mathcal{Y}(K) \) and \( \mathcal{G}(K) \))

1) \(^{30}\) Every sequence \( \{ u^N \} \), \( L^\infty(\Omega, \mathbb{R}^{nm}) \) with \( u^N(t) \in K \ (\forall) t \in \Omega \ \forall \ N \in \mathbb{N} \) admits a weak\(^*\)-convergent subsequence, which generates a generalized control \( \mu \in \mathcal{Y}(K) \).

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\(^{26}\) [Wagner 07D], p. 5, Lemma 2.2. and Theorem 2.4., 2).

\(^{27}\) Cf. [Gamkrelidze 78], pp. 23 ff., and [Müller 99], p. 115 ff.

\(^{28}\) Cf. [Pedregal 97], pp. 96 ff.

\(^{29}\) [Kinderlehrer/Pedregal 91], p. 333, [Müller 99], p. 126, Definition 4.1.

\(^{30}\) [Müller 99], p. 115 f., Theorem 3.1.
2)\textsuperscript{31} Every sequence \( \{ x^N \} \), \( W^{1,\infty}(\Omega, \mathbb{R}^n) \) with \( \| x^N \|_{L^\infty(\Omega, \mathbb{R}^n)} \leq C, J x^N(t) \in K (\forall) t \in \Omega \forall N \in \mathbb{N} \) admits a subsequence \( \{ x^{N'} \} \) with \( x^{N'} \to c \in C^0(\Omega, \mathbb{R}^n) \) and \( J x^{N'} \to L^\infty(\Omega, \mathbb{R}^{nm}) \) \( J x \in L^\infty(\Omega, \mathbb{R}^{nm}) \). Consequently, \( \{ J x^N \} \) generates a generalized gradient control \( \mu \in \mathcal{G}(K) \).

3)\textsuperscript{32} With respect to the topology from Definition 2.3., the set \( \mathcal{Y}(K) \) is sequentially compact.

4)\textsuperscript{33} The subset \( \mathcal{G}(K) \subset \mathcal{Y}(K) \) of the generalized gradient controls is sequentially compact as well.

We close this subsection with a lemma, which is a straightforward conclusion from the mean-value theorem for generalized gradient controls:

\textbf{Lemma 2.7.}\textsuperscript{34} Assume that \( \Omega \subset \mathbb{R}^m \) is the closure of a strongly Lipschitz domain with \( \sigma \in \operatorname{int}(\Omega) \). We consider sequences \( \{ w^N \} \), \( K \) and \( \{ x^N \} \), \( W^{1,\infty}(\Omega, \mathbb{R}^n) \), which satisfy

a) \( w^N \to w \in K \) \( (w^N \text{ and } w \in \mathbb{R}^{nm} \text{ have to be understood as } (n,m)\text{-matrices}) \),

b) \( w^N + J x^N(t) \in K (\forall) t \in \Omega \forall N \in \mathbb{N} \),

c) \( \{ w^N + J x^N \} \) generates a constant generalized gradient control \( \nu = \{ \nu \} \in \mathcal{G}(K) \).

Then it holds: \( w = \left( \begin{array}{c}
\int_K v_{11} \, d\nu(v) \\
\vdots \\
\int_K v_{nm} \, d\nu(v)
\end{array} \right) \) \begin{equation} (2.5) \end{equation}

3. Jensen’s integral inequality for the envelope \( f^{(qc)} \).

a) The third representation theorem for \( f^{(qc)} \).

We arrive at a further representation theorem for \( f^{(qc)} \) if we form the envelope with continuous, “restricted quasiconvex” functions \( g \) instead of lower semicontinuous, quasiconvex functions. The notion of “restricted quasiconvexity” is related to a given convex body \( K \subset \mathbb{R}^{nm} \) again:\textsuperscript{35}

\textbf{Definition 3.1. (Restricted quasiconvex function with values in } \mathbb{R}) \text{ Assume that a function } g: \mathbb{R}^{nm} \to \mathbb{R} \text{ is given such that } \text{dom } (g) \subset \mathbb{R}^{nm} \text{ is a nonempty Borel set, } g \big| \text{dom } (g) \text{ is Borel measurable and bounded from below on every bounded subset of } \text{dom } (g). \text{ The function } g \text{ is called then restricted quasiconvex (with respect to } K) \text{ if it satisfies for all } v \in K \text{ the following restriction of Morrey’s integral inequality:}

\( g(v) = \inf \left\{ \frac{1}{|\Omega|} \int_{\Omega} g(v + J x(t)) \, dt \mid x \in W^{1,\infty}_0(\Omega, \mathbb{R}^n), v + J x(t) \in K (\forall) t \in \Omega \right\} \). \begin{equation} (3.1) \end{equation}

We prove now that \( f^{(qc)} \) may be represented as the envelope of the continuous, restricted quasiconvex functions \( g \).

\textsuperscript{31} [WAGNER 07D], p. 10, Theorem 2.14., 1).

\textsuperscript{32} [BERLIOCCI/LASRY 73], p. 144, Proposition 1, (i); independently proved again in [KRAUT/PICKENHAUS 90], p. 391, Theorem 4.

\textsuperscript{33} [WAGNER 07D], p. 10, Theorem 2.14., 2).

\textsuperscript{34} Ibid., p. 11, Theorem 2.16., 4), applied to \( \mu = \nu = \{ \nu \} \in \mathcal{G}(K) \).

\textsuperscript{35} Our notion differs from KRISTENSEN’S “local quasiconvexity at } \sigma \text{” in the explicit specification of } K. \text{ See [KRISTENSEN 99], p. 4, Definition.
Theorem 3.2. (Third representation theorem for $f^{(gc)}$) Let a function $f \in \mathcal{F}_K$ be given. Then for all $v \in K$, the following representation holds:

$$f^{(gc)}(v) = \sup \left\{ g(v) \mid g : \mathbb{R}^n \to \mathbb{R} \text{ restricted quasiconvex and continuous}, \quad g(v) \leq f(v) \forall v \in \mathbb{R}^n \right\}. \quad (3.2)$$

Proof of Theorem 3.2. • Step 1. Some properties of quasiconvex functions, which may take the value $(+\infty)$. We recall the following assertions:

Lemma 3.3. (Operations with quasiconvex functions with values in $\mathbb{R}$)\(^{36}\)

1) Together with $g_1, g_2 : \mathbb{R}^n \to \mathbb{R}$, any nonnegative linear combination is quasiconvex.
2) Let $v_0 \in \mathbb{R}^n$ and $\mu > 0$ be given. Then the function $h(v) = g(v_0 + \mu v)$ is quasiconvex together with $g(v) : \mathbb{R}^n \to \mathbb{R}$.

Lemma 3.4. (Morrey’s integral inequality for functions with $\text{dom} (g) = K$)\(^{37}\) Let a convex body $K \subset \mathbb{R}^n$ and a function $g : \mathbb{R}^n \to \mathbb{R}$ with $\text{dom} (g) = K$ be given. Assume that $g \mid K$ is measurable and bounded. Then the following assertions hold:

1) For all $v \in \mathbb{R}^n \setminus K$, Morrey’s integral inequality holds in the form $(+\infty) \leq (+\infty)$.
2) $g$ satisfies Morrey’s integral inequality in a point $v \in K$ iff

$$g(v) = \inf \left\{ \frac{1}{|\Omega|} \int_{\Omega} g(v + Jx(t)) \, dt \mid x \in W^{1,\infty}_0 (\Omega, \mathbb{R}^n) \right\}. \quad (3.3)$$

Lemma 3.5. (Rank one convexity and continuity of quasiconvex functions with $\text{dom} (g) = K$)\(^{38}\)

Given a convex body $K \subset \mathbb{R}^n$ and a function $g : \mathbb{R}^n \to \mathbb{R}$ with $\text{dom} (g) = K$. Assume that $g$ is quasiconvex and $g \mid K$ is bounded. Then the restriction $g \mid \text{int} (K)$ is rank one convex and continuous.

• Step 2. Properties of the function $h$. Define the function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(v) = \sup \left\{ g(v) \mid g : \mathbb{R}^n \to \mathbb{R} \text{ restricted quasiconvex and continuous}, \quad g(v) \leq f(v) \forall v \in \mathbb{R}^n \right\}. \quad (3.4)$$

Since the continuous functions

$$g^N(v) = \min_{v \in K} f(v) + N \cdot \text{Dist} (v, K), \quad N \in \mathbb{N}, \quad (\ast)$$

are convex ([ROCKAFFELLAR/WETS 98], p. 51 f., Example 2.25), they are admissible in the forming of $h$, and the inequalities $g^N(v) \leq h(v) \leq f(v)$ for all $v \in \mathbb{R}^n$ imply $\text{dom} (h) = K$. Moreover, all feasible functions in the forming of $h$ are lower semicontinuous, and their epigraphs are closed. Then the epigraph of $h$ as their intersection is closed as well, and $h$ is a lower semicontinuous function. Thus $h$ satisfies conditions 1) and 2) from Definition 1.1. Let $g$ be a function, which is admissible in the forming of $h$. Then for arbitrary $v \in K$ and $x \in W^{1,\infty}_0 (\Omega, \mathbb{R}^n)$ with $v + Jx(t) \in K (\forall t \in \Omega)$, it holds that

$$g(v) \leq \frac{1}{|\Omega|} \int_{\Omega} g(v + Jx(t)) \, dt \leq \frac{1}{|\Omega|} \int_{\Omega} h(v + Jx(t)) \, dt \quad \Rightarrow \quad (3.5)$$

$$h(v) = \sup \left\{ g(v) \mid \ldots \right\} \leq \frac{1}{|\Omega|} \int_{\Omega} h(v + Jx(t)) \, dt.$$
Together with $\text{dom}(h) = K$, we conclude from Lemma 3.4. that $h$ satisfies Morrey’s integral inequality for every $v \in \mathbb{R}^{nm}$. Consequently, $h$ is a quasiconvex function, which is feasible in the forming of $f^{(qc)}$, and we arrive at $h(v) \leq f^{(qc)}(v)$ for all $v \in \mathbb{R}^{nm}$.

**Step 3.** The equation $h(v_0) = f^{(qc)}(v_0)$ for $v_0 \in \text{int}(K)$. Assume on the contrary that $f^{(qc)}(v_0) - h(v_0) = c > 0$ holds in a point $v_0 \in \text{int}(K)$. Referring to the uniform continuity of $f$ on $K$, which is expressed by the $\varepsilon$-$\delta$ relation

$$
|v' - v''| \leq \delta(\varepsilon) < 1 \implies |f(v') - f(v'')| \leq \varepsilon \quad \forall v', v'' \in K,
$$

we choose $0 < \mu < 1$ with the property

$$
|v - (v_0 + \mu(v - v_0))| = |(1 - \mu)(v - v_0)| \leq (1 - \mu) \cdot \text{Diam}(K) \leq \delta(\frac{c}{2}) < 1 \quad \forall v \in K.
$$

Let us define now a function $g : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ by

$$
g(v) = f^{(qc)}(v_0 + \mu(v - v_0)).
$$

It follows that $K = \text{dom}(f^{(qc)}) \subset \text{int}(\text{dom}(g))$, thus, by Lemma 3.3., 2), $g$ is quasiconvex together with $f^{(qc)}$. Consequently, $g$ is continuous at least on $K$ (Lemma 3.5.). For $v \in K$, we obtain further that

$$
f(v_0 + \mu(v - v_0)) - f^{(qc)}(v_0 + \mu(v - v_0)) \geq 0
$$

(this remains true for $(v_0 + \mu(v - v_0)) \in \mathbb{R}^{nm} \setminus K$ as well), from which it follows that

$$
f(v) - g(v) = f(v_0 + \mu(v - v_0)) - f^{(qc)}(v_0 + \mu(v - v_0)) + f(v) - f(v_0 + \mu(v - v_0))
$$

$$
\geq -|f(v) - f(v_0 + \mu(v - v_0))| \geq -\frac{c}{2}.
$$

On the one hand, we arrive at the inequality $f(v) \geq (g(v) - \frac{1}{2}c)$ for all $v \in K$ (and, consequently, for all $v \in \mathbb{R}^{nm}$), but on the other hand, we have

$$
(g(v_0) - \frac{c}{2}) - h(v_0) = f^{(qc)}(v_0) - \frac{c}{2} - h(v_0) = \frac{c}{2} > 0.
$$

This leads to a contradiction since the continuous function $(g(\cdot) - \frac{1}{2}c)|K$ may be extended by Tietze-Urysohn’s theorem ([KURATOWSKI 48], p. 117, Théorème 3) to a continuous function $\tilde{g} : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ on the whole space, which is admissible in the formation of $h$.

**Step 4.** The equation $h(v_0) = f^{(qc)}(v_0)$ for $v_0 \in \partial K$. Assume now that $f^{(qc)}(v_0) - h(v_0) = c > 0$ holds in a point $v_0 \in \partial K$. Using again the uniform continuity of $f$ on $K$, we choose $0 < \mu < 1$ such that at the same time

$$
|v - \mu v| = |(1 - \mu)v| \leq (1 - \mu) \cdot \text{Diam}(K) \leq \delta(\frac{c}{2}) < 1 \quad \forall v \in K
$$

and

$$
|f^{(qc)}(v_0) - f^{(qc)}(\mu v_0)| \leq \frac{c}{4}
$$

hold. The second relation can be achieved with the aid of Theorem 1.3. since the radial limit relation along the ray $R = \bar{\partial} v_0$ implies $f^{(qc)}(v_0) = \lim_{N \rightarrow \infty} f^{(qc)}(v_N)$ for arbitrary sequences $\{v_N\}, R \rightarrow v_0$. Let us define now a function $g : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ as

$$
g(v) = f^{(qc)}(\mu v)
$$
with $K = \text{dom } (f^{qc}) \subseteq \text{int } (\text{dom } (g))$. Again, $g$ is quasiconvex together with $f^{qc}$ (Lemma 3.3., 2)) and continuous at least on $K$ (Lemma 3.5.). For arbitrary $v \in K$, from
\[
f(v) - f^{qc}(v) \geq 0
\] (3.15)

(what remains true for $v \in \mathbb{R}^{nm} \setminus K$ as well) it follows that
\[
f(v) - g(v) = f(v) - f^{qc}(v) + f(v) - f^{qc}(v) \geq -|f(v) - f(v)| \geq -\frac{e}{2}.
\] (3.16)

Consequently, we find $f(v) - (g(v) - \frac{1}{2} c) \geq 0$ for all $v \in K$, but
\[
\left( g(v_0) - \frac{e}{2} \right) - h(v_0) = f^{qc}(v_0) - \frac{e}{2} - f^{qc}(v_0) + f^{qc}(v_0) - h(v_0)
\]
\[
\geq \frac{e}{2} - |f^{qc}(v_0) - f^{qc}(v_0)| \geq \frac{e}{4} > 0.
\] (3.17)

Analogously to Step 3, we arrive at a contradiction since the continuous function $\left( g(\cdot) - \frac{1}{2} c \right) | K$ may be extended as well to a continuous function $\tilde{g} : \mathbb{R}^{nm} \to \mathbb{R}$, which is admissible in the forming of $h$. 

b) The set-valued map $S^{qc}$.

Before turning to the proof of Jensen’s integral inequality, let us recall the properties of the set-valued map $S^{qc}$ from Definition 1.4.

**Theorem 3.6. (Properties of the images $S^{qc}(v)$)** For every $v \in K$, the set $S^{qc}(v) \subseteq \text{rca } ^{pr} (K)$ is nonempty, convex and weak*-sequentially compact.

**Theorem 3.7. (Continuity and semicontinuity properties of the set-valued map $S^{qc}(\cdot)$)**

1) The set-valued map $S^{qc}$ is upper semicontinuous on $K$ and continuous on $\text{int } (K)$.

2) For all $v_0 \in \partial K$, it holds that $S^{qc}(v_0) = \lim_{v \to v_0, v \in \text{rca } ^{pr} (K)} S^{qc}(v)$ where $R = \varnothing v_0$ denotes the ray through $v_0$ starting from the origin.

c) Proof of the Jensen’s integral inequality for $f^{qc}$.

We start with the statement and proof of Jensen’s integral inequality for restricted quasiconvex functions in the sense of Definition 3.1. Throughout the whole section, we assume that $\Omega \subseteq \mathbb{R}^{mn}$ is a strongly Lipschitz domain with $\varnothing \in \text{int } (\Omega)$.

**Theorem 3.8.** Let a convex body $K \subseteq \mathbb{R}^{nm}$ with $\varnothing \in \text{int } (K)$, a function $f \in \mathcal{F}_K$ and the set-valued map $S^{qc} : K \to \mathcal{P} \{ \text{rca } ^{pr} (K) \}$ from Definition 1.4. be given.

1) **(Characterization of the sets $S^{qc}(w)$)** For all $w \in K$ and all $\nu \in \text{rca } ^{pr} (K)$, it holds that
\[
\nu \in S^{qc}(w) \iff g(w) \leq \int_{K} g(v) d\nu(v)
\] (3.18)

for all continuous, restricted quasiconvex functions $g : \mathbb{R}^{nm} \to \mathbb{R}$.

---

39) [WAGNER 07d], p. 16, Theorem 3.4., and p. 21, Theorem 3.9., 1).
40) Ibid., p. 22, Theorem 3.11., 1), and p. 16, Theorem 3.6.
41) Ibid., p. 21, Definition 3.7. and Theorem 3.8., 2).
2) (Jensen’s integral inequality for continuous, restricted quasiconvex functions) For all \( w \in K \) and an arbitrary continuous, restricted quasiconvex function \( g : \mathbb{R}^{nm} \to \mathbb{R} \) it holds that

\[
\int_K g(\begin{pmatrix} \int_K v_{11} \, d\nu(v) & \cdots & \int_K v_{1m} \, d\nu(v) \\
\vdots & \ddots & \vdots \\
\int_K v_{n1} \, d\nu(v) & \cdots & \int_K v_{nm} \, d\nu(v) \end{pmatrix}) \leq \int_K g(\begin{pmatrix} v_{11} & \cdots & v_{1m} \\
\vdots & \ddots & \vdots \\
v_{n1} & \cdots & v_{nm} \end{pmatrix}) \, d\nu(v) \quad \forall \, \nu \in \mathcal{S}^{qc}(w).
\]

Proof of Theorem 3.8. 1) Let \( w \in K \) and \( \nu \in \mathcal{S}^{qc}(w) \) be given. By Definition 1.4., there exist sequences \( \{ \nu^N \}, \text{ int}(K) \) and \( \{ x^N \}, W^1(\Omega, \mathbb{R}^n) \) such that \( w^N + Jx^N(t) \in K \) (\( \forall t \in \Omega \)) and \( \{ w^N + Jx^N \} \) generates the constant generalized control \( \{ \nu \} \). If \( g : \mathbb{R}^{nm} \to \mathbb{R} \) is a continuous, restricted quasiconvex function then for every \( \nu \in \mathcal{S}^{qc} \) it follows that

\[
g(w^N) \leq \frac{1}{|\Omega|} \int_{\Omega} g(w^N + Jx^N(t)) \, dt \implies g(w) = \lim_{N \to \infty} g(w^N) \leq \lim_{N \to \infty} \frac{1}{|\Omega|} \int_{\Omega} g(w^N + Jx^N(t)) \, dt = \int_K g(v) \, d\nu(v).
\]

Fix now a point \( w \in K \). Assume that a measure \( \hat{\nu} \in rca^{pr}(K) \), which satisfies the inequality

\[
g(w) \leq \int_K g(v) \, d\hat{\nu}(v)
\]

for all continuous, restricted quasiconvex functions \( g : \mathbb{R}^{nm} \to \mathbb{R} \), does not belong to \( \mathcal{S}^{qc}(w) \). We equip the space \( rca(K) \cong (C^0(K, \mathbb{R}))^* \) with its weak* topology, getting thus a locally convex space. Observe that, for the weak* topology, the Hausdorff separation axiom is satisfied ([Dunford/Schwartz 88], p. 15, Definition 1): If two measures \( \nu' \), \( \nu'' \in rca(K) \) with \( \nu' \neq \nu'' \) are given then we may find a (norm-)closed ball containing both \( \nu' \) and \( \nu'' \). Since the restriction of the weak* topology to this ball is metrizable ([Dunford/Schwartz 88], p. 426, Theorem 1), there exist disjoint open neighborhoods of \( \nu' \) and \( \nu'' \). Consequently, we may apply the strong separation theorem ([Ioffe/Tichomirov 79], p. 152, Theorem 2) to the disjoint, convex, weak*–compact sets \( \{ \nu \} \) and \( \mathcal{S}^{qc}(w) \). This proves the existence of a linear functional, which is continuous with respect to the weak* topology and strongly separates the both sets. However, the linear, weak*–continuous functionals on \( rca(K) \) are represented precisely through the functions \( f \in C^0(K, \mathbb{R}) \) ([Dunford/Schwartz 88], p. 421, Theorem 9). Thus there exists a function \( f \in C^0(K, \mathbb{R}) \) with

\[
\int_K f(v) \, d\hat{\nu}(v) \leq -\varepsilon < 0 \leq \int_K f(v) \, d\nu(v) \quad \forall \, \nu \in \mathcal{S}^{qc}(w).
\]

Extending the function to \( \mathbb{R}^{nm} \setminus K \) by \( (+\infty) \), we may assume that \( f \) belongs to \( \mathcal{F}_K \) (for the extension, we keep the notation \( f \)). Forming its lower semicontinuous envelope \( f^{qc} \), we obtain with \( f^{qc}(v) \leq f(v) \quad \forall \, v \in K \) ([Wagner 07a], p. 9, Lemma 2.15., 1) and \( \nu \in rca^{pr}(K) \):

\[
\int_K f^{qc}(v) \, d\hat{\nu}(v) \leq \int_K f(v) \, d\hat{\nu}(v),
\]

From the second representation theorem (Theorem 1.5.), we get the implication

\[
0 \leq \int_K f(v) \, d\nu(v) \quad \forall \, \nu \in \mathcal{S}^{qc}(w) \implies 0 \leq f^{qc}(w).
\]

Summing up, we arrive at

\[
\int_K f^{qc}(v) \, d\hat{\nu}(v) \leq -\varepsilon < 0 \leq f^{qc}(w).
\]
By the third representation theorem (Theorem 3.2.), however, there exists a sequence \( \{ g^M \} \) of continuous, restricted quasiconvex functions with \( g^M(v) \leq f^{(qc)}(v) \leq f(v) \) \( \forall v \in K \) and \( \lim_{M \to \infty} g^M(w) = f^{(qc)}(w) \). From our assumption about \( \hat{\nu} \), we obtain

\[
g^M(w) \leq \int_K g^M(v) \, d\hat{\nu}(v) \leq \int_K f^{(qc)}(v) \, d\hat{\nu}(v),
\]

for all \( M \in \mathbb{N} \), and together with (3.25), we arrive at a contradiction:

\[
f^{(qc)}(w) = \lim_{M \to \infty} g^M(w) \leq \int_K f^{(qc)}(v) \, d\hat{\nu}(v) \leq -\varepsilon < 0 \leq f^{(qc)}(w).
\]

It follows that \( \hat{\nu} \) belongs to \( S^{(qc)}(w) \).

2) By Definition 1.4. and Lemma 2.7., it holds that

\[
\nu \in S^{(qc)}(w) \implies w = \left( \int_K v_{11} \, d\nu(v) \quad \cdots \quad \int_K v_{1m} \, d\nu(v) \\
\vdots \quad \vdots \\
\int_K v_{n1} \, d\nu(v) \quad \cdots \quad \int_K v_{nm} \, d\nu(v) \right).
\]

For this reason, the implication “\( \implies \)” from Part 1) may be reformulated as the claimed inequality.

We are now in the position to prove Theorem 1.6.

**Proof of Theorem 1.6.** 1) By Theorem 3.2., \( f^{(qc)} \) admits the representation

\[
f^{(qc)}(v) = \sup \left\{ g(v) \mid g : \mathbb{R}^{nm} \to \mathbb{R} \text{ continuous and restricted quasiconvex}, \quad g(v) \leq f(v) \ \forall v \in \mathbb{R}^{nm} \right\}. \quad (3.29)
\]

Let now \( w \in K \) be given. Then there exists a sequence of continuous, restricted quasiconvex functions \( \{ g^M \} \), which are feasible in the forming of the supremum in (3.29), with \( \lim_{M \to \infty} g^M(w) = f^{(qc)}(w) \) and (obviously) \( g^M(v) \leq f^{(qc)}(v) \) \( \forall v \in K \). To the functions \( g^M \), we may apply Theorem 3.8., 1): If \( \nu \in S^{(qc)}(w) \) then

\[
g^M(w) \leq \int_K g^M(v) \, d\nu(v) \leq \int_K f^{(qc)}(v) \, d\nu(v)
\]

for all \( M \in \mathbb{N} \), and, consequently,

\[
f^{(qc)}(w) = \lim_{M \to \infty} g^M(w) \leq \int_K f^{(qc)}(v) \, d\nu(v).
\]

2) As in the proof of Theorem 3.8., 2), Jensen’s integral inequality is obtained as reformulation of Part 1).

As a corollary of Theorem 1.6., we can describe the support of the measure, which realizes the minimum in the second representation theorem (Theorem 1.5.):

**Theorem 3.9.** (Properties of the measure realizing the minimum in Theorem 1.5.) \(^{42}\) Under the assumptions of Theorem 1.6., let a point \( w \in K \) be given. If a probability measure \( \nu \in S^{(qc)}(w) \) satisfies the equation \( f^{(qc)}(w) = \int_K f(v) \, d\nu(v) \) then \( f^{(qc)}(v) = f(v) \) for \( \nu \text{-almost all } v \in \text{supp}(\nu) \) and for all \( v \in \text{supp}(\nu) \cap \text{int}(K) \).

\(^{42}\) Cf. [BALL/KIRCHHEIM/KRISTENSEN 00], p. 343.
4. Relaxation of (P) by generalized controls.

a) Relaxation of (P) by replacement of the integrand.

Pursuing the first relaxation approach mentioned in the introduction, from Theorem 1.3, the following relaxation theorem for the model problem (P) has been deduced:

**Theorem 4.1. (Relaxation of the problem (P))** 43) Consider the problem (P), assuming that \( m \geq 2 \), \( n \geq 1 \), \( K \subset \mathbb{R}^m \) is an arbitrary convex body with \( \partial \in \text{int}(K) \), and the integrand \( f \) belongs to \( \mathcal{F}_K \).

1) It holds that \( F^{(qc)}(x) = \int_\Omega f^{(qc)}(Jx(t)) \, dt \leq \int_\Omega f(Jx(t)) \, dt = F(x) \) for all admissible functions in (P).

2) For every sequence \( \{x^N\} \) of admissible functions in (P) with \( x^N \rightharpoonup x^N \) \( L^\infty(\Omega, \mathbb{R}^n) \) and \( Jx^N \rightharpoonup J\hat{x} \), the lower semicontinuity relation \( F^{(qc)}(\hat{x}) \leq \liminf _{N \to \infty} F^\#(x^N) \) is satisfied.

3) The minimal values of (P) and of the problem

\[
(P)^{(qc)} \quad F^{(qc)}(x) = \int_\Omega f^{(qc)}(Jx(t)) \, dt \longrightarrow \inf!; \quad x \in W^{1,\infty}_0(\Omega, \mathbb{R}^n); \quad Jx(t) \in K \quad (\forall) \, t \in \Omega
\] 4.1)

coincide. Consequently, \( (P)^{(qc)} \) has the same finite minimal value as problem (P), and every minimizing sequence \( \{x^N\} \) of (P) contains a subsequence \( \{x^{N_k}\} \) converging together with their generalized derivatives weakly* (in the sense of \( L^\infty(\Omega, \mathbb{R}^n) \) resp. \( L^\infty(\Omega, \mathbb{R}^{nm}) \)) to a global minimizer \( \hat{x} \) of \( (P)^{(qc)} \).

---

\[43\] Wagner 07c, p. 3, Theorem 1.3.
b) Relaxation of \((P)\) by generalized controls.

In the following, we will complement Theorem 4.1. by a relaxation theorem involving generalized controls. For this purpose, we introduce into the problems \((P)\) and \((P)^{(qc)}\) formal control variables \(u\). Then the problems may be reformulated as follows:

\[(P)\quad F(x, u) = \int_{\Omega} f(u(t)) \, dt \rightarrow \inf!; \quad (x, u) \in (C^0(\Omega, \mathbb{R}^n) \cap W^{1,p}_0(\Omega, \mathbb{R}^n)) \times L^p(\Omega, \mathbb{R}^{nm});\]

\[G(x, u) = \left( \frac{\partial x_i}{\partial t} (\cdot) - u_{ij}(\cdot) \right)_{j=1, \ldots, m} = \vartheta_{L^p(\Omega, \mathbb{R}^{nm})};\]

\[u \in U = \{ u \in L^p(\Omega, \mathbb{R}^{nm}) \mid u(t) \in K \; (\forall) \; t \in \Omega \}.\]

We take over all assumptions about the data in \((P)\) from Section 1.a). Note that \(1 < p < \infty\) may be chosen since, in consequence of the control restriction \((4.4)\), the feasible state trajectories \(x\) admit Lipschitz continuous representatives for \(1 < p \leq m\) as well. The problem \((P)^{(qc)}\) takes, respectively, the shape

\[(P)^{(qc)}\quad F^{(qc)}(x, u) = \int_{\Omega} f^{(qc)}(u(t)) \, dt \rightarrow \inf!; \quad (x, u) \in (C^0(\Omega, \mathbb{R}^n) \cap W^{1,p}_0(\Omega, \mathbb{R}^n)) \times L^p(\Omega, \mathbb{R}^{nm});\]

\[G(x, u) = \left( \frac{\partial x_i}{\partial t} (\cdot) - u_{ij}(\cdot) \right)_{j=1, \ldots, m} = \vartheta_{L^p(\Omega, \mathbb{R}^{nm})};\]

\[u \in U = \{ u \in L^p(\Omega, \mathbb{R}^{nm}) \mid u(t) \in K \; (\forall) \; t \in \Omega \}.\]

Further, we introduce the following problem involving generalized controls:

\[(\bar{P})^{(qc)}\quad \bar{F}(x, \mu) = \int_{\Omega} \int_K f(v) \, d\mu(v) \, dt \rightarrow \inf!; \quad (x, \mu) \in (C^0(\Omega, \mathbb{R}^n) \cap W^{1,p}_0(\Omega, \mathbb{R}^n)) \times \mathcal{Y}(K);\]

\[\mu_t \in S^{(qc)}(Jx(t)) \; (\forall) \; t \in \Omega.\]

Here \(S^{(qc)}\) is the set-valued map from Definition 1.4. From the differential inclusion \((4.9)\), the relaxed state equation

\[\bar{G}(x, \mu) = \left( \frac{\partial x_i}{\partial t} (\cdot) - \int_K v_{ij} \, d\mu_{ij}(\cdot)(v) \right)_{j=1, \ldots, m} = \vartheta_{L^p(\Omega, \mathbb{R}^{nm})}\]

can be obtained with the same arguments as in the the proof of Theorem 3.8. 2). We are now in the position to show that the minimal values of the problems \((P)^{(qc)}\) and \((\bar{P})^{(qc)}\) are identical. The proof will rely on the second representation theorem for the envelope \(f^{(qc)}\) (Theorem 1.5.), the Jensen’s integral inequality (Theorem 1.6.) as well as SCHÄL’s theorem about the optimal selection (Theorem 5.8.).

**Theorem 4.2. (Coincidence of the minimal values of \((P)^{(qc)}\) and \((\bar{P})^{(qc)}\))** Under the above mentioned assumptions about the data of \((P)\), the problems \((P)^{(qc)}\) and \((\bar{P})^{(qc)}\) admit identical (finite) minimal values.

**Remark.** Note that \((\bar{P})^{(qc)}\) is a problem, whose feasible domain is no longer convex: In spite of the convexity of the sets \(S^{(qc)}(\cdot)\), for \(\lambda, \lambda'' \geq 0\) with \(\lambda + \lambda'' = 1\) and feasible processes \((x', \mu')\), \((x'', \mu'')\) \(\in W^{1,\infty}_0(\Omega, \mathbb{R}^n) \times \mathcal{Y}(K)\) of \((\bar{P})^{(qc)}\), we obtain, in general,

\[\lambda' \mu'_t + \lambda'' \mu''_t \in \lambda' S^{(qc)}(Jx'(t)) + \lambda'' S^{(qc)}(Jx''(t)) \subseteq S^{(qc)}(\lambda' Jx'(t) + \lambda'' Jx''(t)).\]
Proof of Theorem 4.2. Obviously, the minimal values $m^{(qc)}$ resp. $\tilde{m}^{(qc)}$ of the problems $(P)^{(qc)}$ resp. $(\tilde{P})^{(qc)}$ are finite (cf. Theorem 4.1., 3)). Consider a minimizing sequence $\{(x^N, u^N)\}$ for the problem $(P)^{(qc)}$. As a consequence of the first representation theorem for $f^{(qc)}$ (Theorem 1.3.), we have the following lemma:

Lemma 4.3. \(^{44}\) Let functions $f \in \mathcal{F}_K$ and $u \in L^\infty(\Omega, \mathbb{R}^m)$ with $u(t) \in K \ (\forall) t \in \Omega$ be given. Then for arbitrary $\varepsilon > 0$ there exists an index $N_1(\varepsilon) \in \mathbb{N}$ with

$$\left| \int_\Omega f^{(qc)}(1 - \frac{1}{N}) u(t) \, dt - \int_\Omega f^{(qc)}(u(t)) \, dt \right| \leq |\Omega| \varepsilon \quad \forall N \geq N_1(\varepsilon). \quad (4.12)$$

Consequently, we may assume that $u^N(t) \in (1 - 1/N) K \ \forall t \in \Omega \ \forall N \in \mathbb{N}$. Then from the second representation theorem for $f^{(qc)}$ (Theorem 1.5.), we conclude that

$$F^{(qc)}(x^N, u^N) = \int_\Omega f^{(qc)}(u^N(t)) \, dt = \int_\Omega \left( \inf_{\nu \in S^{(qc)}(u^N(t))} \int_K f(v) \, dv(\nu) \right) \, dt. \quad (4.13)$$

Let us check now whether SCHÄL’s theorem (Theorem 5.8.) can be applied to $\Omega \subset \mathbb{R}^m$, the compact, separable metric space $rca^{pr}(K)$ (cf. Theorem 2.2., 3)), the function $g(\nu) = (f, \nu)$ and the set-valued map $t \mapsto S^{(qc)}(u^N(t))$. By Theorem 3.6., the sets $S^{(qc)}(u^N(t))$ are nonempty and weak*-sequentially compact. Thus on each of the sets $S^{(qc)}(u^N(t))$, the weak*-continuous linear functional $g(\nu)$ takes on its minimal value. By [ELSTRODT 96], p. 108, Corollary 4.14., a), the measurable function $u^N$ may be approximated in uniform convergence by simple functions $u^{N,K}$, which are adapted to Lebesgue subsets of $\Omega$. Since $u^N(t) \in (1 - 1/N) K \ \forall t \in \Omega \ \forall N \in \mathbb{N}$, we may further assume that $u^{N,K}(t) \in (1 - 1/(2N)) K \ \forall t \in \Omega \ \forall N, K \in \mathbb{N}$. Define now the set-valued maps $S^{N,K}: \Omega \to \mathcal{F}(rca^{pr}(K))$ by

$$S^{N,K}(t) = \left\{ \nu' \in rca^{pr}(K) \mid \exists \nu'' \in S^{(qc)}(u^{N,K}(t)) \text{ with } \sigma(\nu', \nu'') \leq \frac{1}{K} \right\}. \quad (4.14)$$

We claim first that the maps $S^{N,K}$ are measurable. Obviously, the images $S^{N,K}(t)$ are nonempty and weak*-closed. Choose now an arbitrary weak*-closed set $E \subseteq rca^{pr}(K)$ and consider its inverse image

$$\left\{ t \in \Omega \mid S^{N,K}(t) \cap E \neq \emptyset \right\} = \left\{ t \in \Omega \mid \exists \nu' \in E \exists \nu'' \in S^{(qc)}(u^{N,K}(t)) : \sigma(\nu', \nu'') \leq \frac{1}{K} \right\}. \quad (4.15)$$

From the definition of the functions $u^{N,K}$, we obtain for all $t \in \Omega$

$$u^{N,K}(t) = \sum_s 1_{A^{N,K}_s}(t) \cdot u^{N,K}_s \quad \Rightarrow \quad S^{(qc)}(u^{N,K}(t)) = \sum_s 1_{A^{N,K}_s}(t) \cdot S^{(qc)}(u^{N,K}_s), \quad (4.16)$$

what further implies that

$$\left\{ t \in \Omega \mid S^{N,K}(t) \cap E \neq \emptyset \right\}$$

$$= \bigcup_s \left\{ t \in A^{N,K}_s \mid \exists \nu' \in E \exists \nu'' \in S^{(qc)}(u^{N,K}_s) : \sigma(\nu', \nu'') \leq \frac{1}{K} \right\} \quad (4.17)$$

$$= \bigcup_s \left\{ t \in A^{N,K}_s \mid \text{Dist}(E, S^{(qc)}(u^{N,K}_s)) \leq \frac{1}{K} \right\}. \quad (4.18)$$

Together with the sets $A^{N,K}_s$, the set in (4.18) is a Lebesgue set, and the set-valued maps $S^{N,K}$ are measurable. Moreover, $S^{N,K}(t)$ admits the representation

$$S^{N,K}(t) = \bigcup_{\nu \in S^{(qc)}(u^{N,K}(t))} K(\nu, \frac{1}{K}) \quad (4.19)$$

\(^{44}\) [WAGNER 07c], p. 12, Lemma 3.6.
with (norm-)closed balls $K(\nu,1/K) \subseteq \text{rca}^{pr}(K)$ (which are weak*-closed as well). Since in each of these balls, its intersection with the countable subset of $\text{rca}^{pr}(K)$ from Theorem 2.2., 3) lies dense, the set-valued maps $S^{N,K}$ are separable as well. Finally, we get from (4.14):

$$\mathcal{H}\left(S^{N,K}(t), S^{(qc)}(u^{N,K}(t))\right) \to 0 \quad (4.20)$$

for all $t \in \Omega$, and from $u^{N,K}(t), u^N(t) \in (1 - 1/(2N)) K \ \forall t \in \Omega \ \forall N, K \in \mathbb{N}$, the uniform convergence $u^{N,K} \to u^N$ and the continuity of the set-valued map $S^{(qc)}$ on int $(K)$ (Theorem 3.7., 1), we obtain

$$\mathcal{H}\left(S^{(qc)}(u^{N,K}(t)), S^{(qc)}(u^N(t))\right) \to 0 \quad (4.21)$$

for all $t \in \Omega$ as well. Summing up, we arrive at

$$\lim_{K \to \infty} S^{N,K}(t) = S^{(qc)}(u^N(t)) \quad (4.22)$$

for all $t \in \Omega$ and all $N \in \mathbb{N}$, and the application of Theorem 5.8. is justified. As a consequence, we find a measurable selection, i.e. a generalized control $\mu^N = \{ \mu^N_t \} \in \mathcal{Y}(K)$ with $\mu^N_t \in S^{(qc)}(u^N(t))$ for all $t \in \Omega$, which satisfies

$$\int_K f(v) d\mu^N(v) = \inf \left\{ \int_K f(v) d\nu(v) \mid \nu \in S^{(qc)}(u^N(t)) \right\} \quad (4.23)$$

for all $t \in \Omega$. Thus the pairs $(x^N, \mu^N)$ are feasible elements in $(\tilde{P})^{(qc)}$, and it follows that

$$F^{(qc)}(x^N, u^N) = \int_{\Omega} \inf_{\nu \in S^{(qc)}(u^N(t))} \int_K f(v) d\nu(v) \ dt \quad (4.24)$$

$$= \int_{\Omega} \int_K f(v) d\mu^N_t(v) dt = \tilde{F}(x^N, \mu^N) \geq \tilde{m}^{(qc)};$$

consequently, the inequality $m^{(qc)} \geq \tilde{m}^{(qc)}$ holds. Conversely, let a minimizing sequence $\{ (x^N, \mu^N) \}$ for $(\tilde{P})^{(qc)}$ be given. Since $\mu^N_t \in S^{(qc)}(J x^N(t)) \ \forall t \in \Omega \ \forall N \in \mathbb{N}$ and $f(v) \geq f^{(qc)}(v) \ \forall v \in K$, we conclude with the Jensen’s integral inequality (Theorem 1.6.) that

$$\tilde{F}(x^N, \mu^N) = \int_{\Omega} \int_K f(v) d\mu^N_t(v) dt$$

$$\geq \int_{\Omega} \int_K f^{(qc)}(v) d\mu^N_t(v) dt \geq \int_{\Omega} f^{(qc)}(J x^N(t)) dt = F^{(qc)}(x^N(t), J x^N(t)) \geq m^{(qc)}.$$ 

Thus the reverse inequality $\tilde{m}^{(qc)} \geq m^{(qc)}$ holds as well, and the proof is complete. ■

5. Appendix: Set-valued maps.

a) Painlevé-Kuratowski limits.

Definition 5.1. (Painlevé-Kuratowski limits for set sequences)\(^{(45)}\) Within a metric space $[X, \sigma]$, let a sequence of sets $\{E_N\}, \mathcal{F}(X)$ be given. We define

$$\liminf_{N \to \infty} E_N = \{ \nu \in X \mid \exists \{ \nu^N \}, X \text{ with } \nu^N \in E_N \text{ for almost all } N \text{ and } \lim_{N \to \infty} \sigma(\nu^N, \nu) = 0 \};$$

$$\limsup_{N \to \infty} E_N = \{ \nu \in X \mid \exists \{ \nu^N \}, X \text{ with } \nu^N \in E_N \text{ for infinitely many } N \text{ and } \lim_{N \to \infty} \sigma(\nu^N, \nu) = 0 \};$$

$$\lim_{N \to \infty} E_N = E \iff \liminf_{N \to \infty} E_N = \limsup_{N \to \infty} E_N = E. \quad (5.1)$$

\(^{(45)}\) [Aubin/Frankowska 90], p. 17, Definition 1.1.1.; see also [Rockafellar/Wets 98], p. 109, Definition 4.1.
In this definition, “for almost all N” means “except at most finitely many”.

**Definition 5.2. (Hausdorff distance in the metric space X)**\(^{46}\) Let \([X, \sigma]\) be a compact metric space. Then we define the Hausdorff distance of nonempty, closed subsets \(S', S'' \subseteq X\) by

\[
\mathcal{H}(S', S'') = \max \left( \max_{\nu' \in S'} \text{Dist}(\nu', S''), \ max_{\nu'' \in S''} \text{Dist}(\nu'', S') \right) \quad \text{resp.} \tag{5.4}
\]

\(\mathcal{H}(S', S'') \leq \varepsilon \iff \text{for every } \nu' \in S' \text{ there exists } \nu'' \in S'' \text{ with } \sigma(\nu', \nu'') \leq \varepsilon, \text{ and for every } \nu'' \in S'' \text{ there exists } \nu' \in S' \text{ with } \sigma(\nu', \nu'') \leq \varepsilon.\)

**Definition 5.3. (Painlevé-Kuratowski limits for set-valued maps)**\(^{47}\) Let a nonempty, compact subset \(K \subseteq \mathbb{R}^{nm}\) with \(o \in \text{int}(K)\) and a compact metric space \([X, \sigma]\) be given. We consider a set-valued map \(S : K \to \mathcal{P}(X)\) with nonempty, closed images, and define for \(v_0 \in K\)

\[
\liminf_{v \to v_0}^K S(v) = \bigcap_{v^N \to v_0} \liminf_{N \to \infty}^K S(v^N) \tag{5.5}
\]

\[
\limsup_{v \to v_0}^K S(v) = \bigcup_{v^N \to v_0} \limsup_{N \to \infty}^K S(v^N) \tag{5.6}
\]

\[
\lim_{v \to v_0}^K S(v) = E \iff \liminf_{v \to v_0}^K S(v) = \limsup_{v \to v_0}^K S(v) = E. \tag{5.7}
\]

(b) Semicontinuous and continuous set-valued maps.

**Definition 5.4. (Semicontinuity and continuity of set-valued maps)**\(^{48}\) Let a nonempty, compact set \(K \subseteq \mathbb{R}^{nm}\) with \(o \in \text{int}(K)\) and a compact metric space \([X, \sigma]\) be given. We consider a set-valued map \(S : K \to \mathcal{P}(X)\) with nonempty, closed images.

1) The set-valued map \(S\) is called lower semicontinuous in \(v_0 \in K\) if \(S(v_0) \subseteq \liminf_{v \to v_0}^K S(v)\) holds.
2) The set-valued map \(S\) is called upper semicontinuous in \(v_0 \in K\) if \(\limsup_{v \to v_0}^K S(v) \subseteq S(v_0)\) holds.
3) The set-valued map \(S\) is called continuous in \(v_0 \in K\) if \(S(v_0) = \lim_{v \to v_0}^K S(v)\) holds.

(c) Measurable set-valued maps and the theorem about the optimal selection.

**Definition 5.5. (Measurable set-valued maps)**\(^{49}\) Assume that \(\Omega \subseteq \mathbb{R}^n\) as the closure of a bounded domain and a compact, separable metric space \([X, \sigma]\) are given. A set-valued map \(S : \Omega \to \mathcal{P}(X)\) with nonempty, closed images is called (Lebesgue) measurable if for every closed subset \(E \subseteq X\) the inverse image \(\{ t \in \Omega \mid S(t) \cap E \neq \emptyset \} \subset \mathbb{R}^n\) is a Lebesgue set.

Note that in the following theorem, the notion of measurability has to be related to the \(\sigma\)-algebra of the Lebesgue subsets of \(\Omega\) (cf. \[AUBIN/Frankowska 90\], p. 321).

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\(^{46}\) [ROCKAFELLAR/WETS 98], p. 117, Example 4.13.

\(^{47}\) [AUBIN/Frankowska 90], p. 41, Definition 1.4.6.; see also [ROCKAFELLAR/WETS 98], p. 152.

\(^{48}\) [AUBIN/Frankowska 90], p. 39 f., Definitions 1.4.2. and 1.4.3.; see also [ROCKAFELLAR/WETS 98], p. 152, Definition 5.4.

\(^{49}\) [AUBIN/Frankowska 90], p. 307 f., Remark.
**Theorem 5.6. (Sequences of measurable set-valued maps)** Assume that $\Omega \subset \mathbb{R}^m$ as the closure of a bounded domain and a compact, separable metric space $[X, \sigma]$ are given. Consider a sequence of measurable set-valued maps $S^N : \Omega \to \mathcal{P}(X)$ such that the Painlevé-Kuratowski limit $S(t) = \lim_{K \to \infty} S^N(t)$ exists for all $t \in \Omega$. Then $S : \Omega \to \mathcal{P}(X)$ is a measurable set-valued map as well.

**Definition 5.7. (Separable set-valued maps)** Assume that $\mathcal{S} \subset \mathbb{R}^m$ as the closure of a bounded domain and a compact, separable metric space $[X, \sigma]$ with the countable, dense subset $\tilde{X}$ are given. A set-valued map $S : \Omega \to \mathcal{P}(X)$ with nonempty, closed images is called separable if it is measurable, and the intersections $S(t) \cap \tilde{X}$ are dense in $S(t)$ for all $t \in \Omega$.

The following theorem may be understood as a generalization of the well-known Filippov lemma:

**Theorem 5.8. (Measurability of the optimal selection)** Assume that $\Omega \subset \mathbb{R}^m$ as the closure of a bounded domain and a compact, separable metric space $[X, \sigma]$ with the countable, dense subset $\tilde{X}$ are given. Consider a function $g(t, \nu) : \Omega \times X \to \mathbb{R}$ and a set-valued map $S : \Omega \to \mathcal{P}(X)$ with nonempty, closed images, which satisfy the following assumptions:

a) $g(t, \nu)$ is a Carathéodory function; consequently, $g$ is (Lebesgue) measurable with respect to $t$ and continuous with respect to $\nu$;

b) for every $t \in \Omega$ there exists an “optimal” element $\hat{\nu} \in S(t)$ with $g(t, \hat{\nu}) = \inf \{ g(t, \nu) \mid \nu \in S(t) \}$;

c) the set-valued map $S$ admits an approximation by a sequence of separable set-valued maps $S^N : \Omega \to \mathcal{P}(X)$ with $\lim_{K \to \infty} S^N(t) = S(t)$ for all $t \in \Omega$.

Then there exists a Lebesgue measurable function $h : \Omega \to X$ (an “optimal selection”) with

$$h(t) \in S(t) \quad \text{and} \quad g(t, h(t)) = \inf \{ g(t, \nu) \mid \nu \in S(t) \}.$$  \hspace{1cm} (5.8)

for all $t \in \Omega$.

**References.**


50) [Aubin/Frankowska 90], p. 312 f., Theorem 8.2.5.

51) [Schäl 74], p. 219, with $[S, \Theta] = [\Omega, \mathcal{L}_\Omega]$.

52) Cf. [Aubin/Frankowska 90], p. 316, Theorem 8.2.10.

53) [Schäl 74], p. 220, Theorem 3.
32. [Rockafellar/Wets 98] Rockafellar, R. T.; Wets, R. J.-B.: Variational Analysis. Springer; Berlin etc. 1998 (Grundlehren 317)

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