Pontryagin's maximum principle for multidimensional control problems in image processing. Updated version

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1. Introduction.

a) Multidimensional control problems of Dieudonné-Rashevsky type.

In the present paper, we investigate optimal control problems involving first-order PDE's together with boundary conditions, phase and control restrictions. Following CESARI, problems of this type are called *Dieudonné-Rashevsky type problems.* [01] In the simplest case, a Dieudonné-Rashevsky type problem will be obtained when adding to the basic problem of multidimensional calculus of variations,

$$(\mathrm{V})_0 \quad F(x) \, = \, \int_\Omega f(t, x(t), Jx(t)) \, dt \; \longrightarrow \; \inf!\,; \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,, \quad \Omega \subset \mathbb{R}^m\,, \tag{1.1}$$

restrictions for the partial derivatives of x. Imposing, for example, in $(\mathrm{V})_0$ the additional condition

$$Jx(t) \, = \, \begin{pmatrix} \dfrac{\partial x_1}{\partial t_1}(t) & \dots & \dfrac{\partial x_1}{\partial t_m}(t) \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial t_1}(t) & \dots & \dfrac{\partial x_n}{\partial t_m}(t) \end{pmatrix} \in \mathrm{K} \subset \mathbb{R}^{nm} \;\; (\forall)\, t \in \Omega\,, \tag{1.2}$$

for the Jacobian of x, we get the optimal control problem

$$(\mathrm{P})_0 \quad F(x,u) = \int_\Omega f(t, x(t), u(t)) \, dt \; \longrightarrow \; \inf!\,; \quad (x,u) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \, \times \, L^p(\Omega, \mathbb{R}^{nm})\,; \tag{1.3}$$

$$Jx(t) \, = \, u(t) \;\; (\forall)\, t \in \Omega\,; \tag{1.4}$$

$$u \in \mathrm{U} = \big\{ \, u \in L^p(\Omega, \mathbb{R}^{nm}) \; \big| \; u(t) \in \mathrm{K} \;\; (\forall)\, t \in \Omega \, \big\}\,. \tag{1.5}$$

Until now, torsion problems for prismatic bars with elastic or elastic-plastic material laws were considered as the classical field of application for Dieudonné-Rashevsky type problems. [02] In these problems, the state variable is a stress function T while the shear stresses τ_{zx} and τ_{zy} within the cross-sectional plane of the bar (which are proportional to the partial derivatives of T) act as controls. [03] Then for materials like steel, a control restriction results from the fact that the modulus of the resulting shear stress is not allowed to exceed the so-called yield point. [04] Further instances are optimization problems for convex bodies, e. g. maximization of the surface for given width and diameter, while the bodies are described through support functions in

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[01] [CESARI 69] , p. 339: "problem ... with differential equations ... written in the Dieudonné-Rashevsky form".

[02] In the case of elastic material law, St.-Venant's torsion has already been described through a variational problem in [FUNK 62] , pp. 531 ff., and through an optimal control problem in [LUR'E 75] , pp. 240 ff. and [WAGNER 96] , pp. 76 ff. Using the energy functional [SAUER 80] , p. 20, (4.-60), it is possible to identify in this framework the warping torsion as well. Torsion problems with elastic-plastic material law have been investigated by TING (see, for instance, [TING 69A] , p. 531 f., and [TING 69B]).

[03] [SAUER 80] , pp. 8 – 20.

[04] [CHMELKA/MELAN 76] , pp. 38 – 45.

spherical coordinates. These lead again to convex Dieudonné-Rashevsky type problems with linear state equations. [05)]

Starting from a completely different viewpoint, DACOROGNA and MARCELLINI arrived at Dieudonné-Rashevsky type problems in their papers on underdetermined boundary-value problems for nonlinear first-order PDE's at the end of the 90s. [06)] Assuming e. g. that a function $f(t,\xi,v)\colon\ \Omega\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ satisfies the compatibility condition $f(t,\mathfrak{o},\mathfrak{o})\leqslant 0\ (\forall)\,t\in\Omega\subset\mathbb{R}^m$ together with certain coercivity and convexity conditions, the Dirichlet boundary-value problem

$$f(t,x(t),\nabla x(t))\ =\ 0\quad(\forall)\,t\in\Omega\,,\quad x\in W_0^{1,\infty}(\Omega,\mathbb{R}) \tag{1.6}$$

is equivalent to the control problem

$$(\mathrm{P})_1\quad F(x,u)=\int_\Omega\big(\,f(t,x(t),u(t))\,\big)^2\,dt\ \longrightarrow\ \inf!\,;\quad (x,u)\in W_0^{1,\infty}(\Omega,\mathbb{R})\ \times\ L^\infty(\Omega,\mathbb{R}^m)\,; \tag{1.7}$$

$$\nabla x(t)\ =\ u(t)\ \ (\forall)\,t\in\Omega\,; \tag{1.8}$$

$$u\in\mathrm{U}=\big\{\,u\in L^\infty(\Omega,\mathbb{R}^m)\ \big|\ u(t)\in\mathrm{K}\ \ (\forall)\,t\in\Omega\,\big\} \tag{1.9}$$

where the set $\mathrm{K}\subset\mathbb{R}^m$, to be constructed on the base of f, is a convex body. Moreover, the set of global minimizers is uncountable and dense in the set $\{\,x\in W_0^{1,\infty}(\Omega,\mathbb{R})\ \big|\ f(t,x(t),\nabla x(t))\leqslant 0\ \ (\forall)\,t\in\Omega\,\}$ with respect to the L^∞-norm topology. [07)] Even if the assumptions formulated by DACOROGNA/MARCELLINI are not (or not completely) satisfied, it seems plausible to treat ill-posed boundary-value problems of the shape (1.6) or, more generally,

$$f(t,x(t),Jx(t))\ =\ 0\quad(\forall)\,t\in\Omega\,,\quad x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\quad n>1 \tag{1.10}$$

as multidimensional control problems. Besides the defect minimization term, within the objective one could consider a regularization term as well while the control restriction acts as a parameter.

Although Dieudonné-Rashevsky type problems are less intensely investigated than control problems with second-order PDE's, [08)] existence and relaxation theorems as well as first-order necessary optimality conditions in the form of the so-called ε-maximum principle [09)] and duality theorems [10)] have been available since the 90s for convex problems with linear state equations ($G(t,x(t),Jx(t),u(t))\ =\ Jx(t)-u(t)\ =\ 0$ resp. $G(\,\dots)=Jx(t)-A(t)\,x(t)-B(t)\,u(t)=0$). The corresponding theory was developed substantially by KLÖTZLER, PICKENHAIN and the author.

b) Aims and outline of the paper.

In the present paper, we pursue two purposes. Thereby we restrict ourselves to the investigation of convex problems, while the quasiconvex case will be treated in subsequent publications. Our first goal is to eliminate

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05) [ANDREJEWA/KLÖTZLER 84A] and [ANDREJEWA/KLÖTZLER 84B], p. 149 f.

06) [DACOROGNA/MARCELLINI 97], [DACOROGNA/MARCELLINI 98] and [DACOROGNA/MARCELLINI 99].

07) This follows from [DACOROGNA/MARCELLINI 99], p. 35, Theorem 2.3., and the analysis of the related proof; ibid., pp. 43 – 47.

08) We refer e. g. to [LIONS 71] and [TRÖLTZSCH 05A].

09) For a survey of the papers concerning necessary conditions, we refer to Section 2.a) below.

10) Analogous to the field theory in the calculus of variations, duality theorems and saddle-point conditions have been proved. Among the papers concerning duality, we mention [KLÖTZLER 95], [KLÖTZLER 98], [PICKENHAIN 91], pp. 26 – 94, [PICKENHAIN 92B], [PICKENHAIN 02], [PICKENHAIN/TAMMER 91], [PICKENHAIN/WAGNER 01] and [WAGNER 00] as well as the seminal paper [KLÖTZLER 79].

the hitherto existing dependence on the artificial parameter ε within the first-order necessary conditions (which are cited below as Theorem 2.1.). To this end, we provide a new proof for Pontryagin's principle which results in a substantial improvement of the first-order necessary optimality conditions (Theorems 2.2., 2.3. and 3.3.).

Our second purpose is to introduce Dieudonné-Rashevsky type problems into mathematical image processing. We consider two problems. The first one is the conversion of time-dependent given image data into the so-called *optical flow*, the second one is the reconstruction of a piece of the earth's surface from aerial photographs (*Shape from Shading*). A common way to treat these problems is their reformulation as problems of multidimensional calculus of variations. Both problems have in common that the objectives consist of two terms where the first one minimizes the defect in the underlying equation while the second one is a regularization term. In the case of the optical flow problem, the correspondence between different regularization terms and the interpretation of the Euler-Lagrange equations as diffusion equations is well understood. [11] In the literature, the advantages and disadvantages of the numerous regularization terms have been widely discussed. In particular, regularization terms with convex and nonquadratic integrands have been well investigated. [12] We argue that in both problems, the addition of control restrictions to the variational problems is reasonable. [13] [14] In the case of Shape from Shading this is even mandatory since, actually, the situation described by Dacorogna/Marcellini's theory occurs (Theorem 5.1.). Depending on the formulation, we arrive at Dieudonné-Rashevsky type problems which have to be relaxed either convex or quasiconvex. In the convex case, the newly formulated necessary conditions are now available.

The paper is organized as follows. This section will close with a collection of notations (Section 1.c)). In Section 2, we start with a survey on the known necessary optimality conditions for convex Dieudonné-Rashevsky type problems (Section 2.a)). Subsequently, we rephrase and prove Pontryagin's maximum principle in its parameter-free version (Section 2.b)). Section 3 is devoted to the Helmholtz-Weyl decomposition of the multipliers into "gradient" and "curl" components. Then we present the application problems from image processing. Section 4 is addressed to the optical flow problem, Section 5 to the Shape from Shading problem. In both cases, we give a survey of the previous treatment of the problems within the framework of calculus of variations and discuss their reformulation as multidimensional control problems. In the convex case, we state the necessary optimality conditions (Theorems 4.1., 4.2. and 5.3.).

c) Notation.

Let $k \in \{\, 0,\, 1,\, ,\, \dots\, ,\, \infty \,\}$ and $1 \leqslant p \leqslant \infty$. Then $C^k(\Omega, \mathbb{R}^r)$, $L^p(\Omega, \mathbb{R}^r)$ and $W^{k,p}(\Omega, \mathbb{R}^r)$ denote the spaces of r-dimensional vector functions whose components are k-times continuously differentiable, resp. belong to $L^p(\Omega)$ or to the Sobolev space of $L^p(\Omega)$-functions with weak derivatives up to kth order in $L^p(\Omega)$. In addition, functions within the subspaces $C_0^k(\Omega, \mathbb{R}^r) \subset C^k(\Omega, \mathbb{R}^r)$ resp. $W_0^{k,p}(\Omega, \mathbb{R}^r) \subset W_p^{k,p}(\Omega, \mathbb{R}^r)$ are compactly supported. The subspaces $G^p(\Omega, \mathbb{R}^{nm})$, $R^p(\Omega, \mathbb{R}^{nm}) \subset L^p(\Omega, \mathbb{R}^{nm})$ of "gradient" and "curl" fields will be defined in Section 3.a) below. The symbols x_{t_j} and $\partial x / \partial t_j$ may denote the classical as well as the weak partial derivative of x by t_j. Likewise, ∇x and Jx denote the (classical or weak) gradient resp. Jacobian

[11] Cf. [Weickert/Brox 02] and [Weickert/Schnörr 01].

[12] Recently nonconvex integrands are included in these considerations as well, see [Aubert/Kornprobst 02], pp. 80 ff. and pp. 187 ff., [Chipot/March/Vitulano 01]. In the case of the Shape from Shading problem, this leads to a convex relaxation, and in the case of the optical flow problem to a quasiconvex relaxation of the variational problems.

[13] First numerical experiments for the optical flow problem with control restrictions look very promising but are not included here. See a forthcoming paper together with C. Brune.

[14] The problem of *image restoration* resp. *image smoothing* allows a reformulation as Dieudonné-Rashevsky type problem as well, see [Wagner 2006], pp. 108 – 112.

of x. $\mathbb{1}_{\mathrm{A}}: \mathbb{R}^r \to \mathbb{R}$ with $\mathbb{1}_{\mathrm{A}}(t) = 1 \iff t \in \mathrm{A}$ and $\mathbb{1}_{\mathrm{A}}(t) = 0 \iff t \notin \mathrm{A}$ denotes the characteristic function of the set $\mathrm{A} \subseteq \mathbb{R}^r$. $\mathrm{int}\,(\mathrm{A})$, $\mathrm{cl}\,(\mathrm{A})$, $\partial\mathrm{A}$, $\mathrm{cl\,cone}\,(\mathrm{A})$ and $|\,\mathrm{A}\,|$ denote the interior, closure, boundary, the closed positive hull and the r-dimensional Lebesgue measure of the set $\mathrm{A} \subseteq \mathbb{R}^r$, respectively. Finally, we explain three nonstandard notations: $\{\, x^N \,\},\ \mathrm{A}$ denotes a sequence $\{\, x^N \,\}$ with members $x^N \in \mathrm{A}$. If $\mathrm{A} \subseteq \mathbb{R}^r$ then the abbreviation "$(\forall)\, t \in \mathrm{A}$" has to be read as "for almost all $t \in \mathrm{A}$" resp. "for all $t \in \mathrm{A}$ except some r-dimensional Lebesgue null set". The symbol $\mathfrak{o}$ denotes, dependent on the context, the zero element resp. the zero function of the underlying space.

2. Pontryagin's maximum principle for Dieudonné-Rashevsky type problems.

a) The ε-maximum principle.

We consider multidimensional control problems of the shape

$$(\mathrm{P})_0 \quad F(x,u) = \int_\Omega f(t, x(t), u(t))\, dt \longrightarrow \inf!\,; \quad (x,u) \in \big(C^0(\Omega, \mathbb{R}^n) \cap W_0^{1,p}(\Omega, \mathbb{R}^n) \big) \times L^p(\Omega, \mathbb{R}^{nm})\,; \tag{2.1}$$

$$G(x,u) = \Big(\frac{\partial x_i}{\partial t_j}(\,\cdot\,) - u_{ij}(\,\cdot\,) \Big)_{\substack{i=1,\dots,n \\ j=1,\dots,m}} = \mathfrak{o}_{L^p(\Omega, \mathbb{R}^{nm})}\,; \tag{2.2}$$

$$u \in \mathrm{U} = \big\{\, u \in L^p(\Omega, \mathbb{R}^{nm}) \,\big|\, u(t) \in \mathrm{K}\ \ (\forall)\, t \in \Omega \,\big\}\,, \tag{2.3}$$

and throughout the section make the following *assumptions about the data of* $(\mathrm{P})_0$: Let $n \geqslant 1$, $m \geqslant 2$ and $1 < p < \infty$. $\Omega \subset \mathbb{R}^m$ is the closure of a bounded Lipschitz domain. [15] The function $f(t, \xi, v): \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}$ is measurable and essentially bounded with respect to t and continuously differentiable with respect to all ξ_i and v_{ij}. $\mathrm{K} \subset \mathbb{R}^{nm}$ is a convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. These assumptions guarantee the existence of a feasible solution (the zero solution). For any feasible solution (x,u) of $(\mathrm{P})_0$ it follows that $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n) \cap W_0^{1,p}(\Omega, \mathbb{R}^n)$ since $Jx(t) \in \mathrm{K}\ (\forall)\, t \in \Omega$. x thus admits a Lipschitz representative in the case $1 < p \leqslant m$ as well. [16]

The general theorems of Ioffe/Tichomirow and Ginsburg/Ioffe [17] ("Lagrange's principle") cannot be applied to this problem since its assumptions about the operator $G(x,u): W_0^{1,p}(\Omega, \mathbb{R}^n) \times \mathrm{U} \to L^p(\Omega, \mathbb{R}^{nm})$ fail: the range

$$\big\{\, G_x(x^*, u^*)\,(x,u) \,\big|\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n) \,\big\} = \big\{\, z \in L^p(\Omega, \mathbb{R}^{nm}) \,\big|\, \exists\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n) \text{ with } z = Jx \,\big\} \tag{2.4}$$

has infinite codimension in $L^p(\Omega, \mathbb{R}^{nm})$ (cf. Section 3.a) below). Consequently, the subspace of $L^p(\Omega, \mathbb{R}^{nm})$ generated by the feasible controls u violating the integrability conditions [18]

$$\int_\Omega \Big(u_{ij}(t)\, \frac{\partial \psi}{\partial t_k}(t) - u_{ik}(t)\, \frac{\partial \psi}{\partial t_j}(t) \Big)\, dt = 0 \quad \forall\, \psi \in C_0^\infty(\Omega, \mathbb{R})\,,\ 1 \leqslant i \leqslant n\,,\ 1 \leqslant j, k \leqslant m \tag{2.5}$$

is infinite-dimensional. For this reason, the proof of the Pontryagin maximum principle for control problems with single integrals cannot be carried over to $(\mathrm{P})_0$. First-order necessary conditions for Dieudonné-

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15) [Evans/Gariepy 92], p. 127.

16) Ibid., p. 131, Theorem 5.

17) [Ioffe/Tichomirow 79], p. 73 ff., Theorem 3, resp. [Ginsburg/Ioffe 96], p. 92, Theorem 3.3., and p. 96, Theorem 3.6.

18) Cf. [Wagner 99], p. 169 f., Theorem 1.4.

Rashevsky type problems were proven at first by CESARI, RUND and KLÖTZLER, under assumptions, however, which nearly exclude these conditions from practical application. [19] Since the 90s, KLÖTZLER, PICKENHAIN and WAGNER have pursued a different approach leading to the so-called ε-maximum principle. In this set of first-order necessary optimality conditions, the multipliers as well as the conditions themselves depend on an additional parameter $\varepsilon > 0$. [20] This theorem, assuming explicitly the convexity of the integrand with respect to v, reads as follows:

Theorem 2.1. (ε-maximum principle for $(\mathrm{P})_0$ in the smooth-convex case) [21] *Consider the problem $(\mathrm{P})_0$ under the above mentioned assumptions about the data. In addition, assume that $f(t,\xi,v)$ is continuous in t and convex with respect to v for all fixed $(t,\xi) \in \Omega \times \mathbb{R}^n$. If (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$ then for any parameter $\varepsilon > 0$ there exist multipliers $\lambda^\varepsilon > 0$ and $y^\varepsilon \in L^q(\Omega, \mathbb{R}^{nm})$, $p^{-1} + q^{-1} = 1$, satisfying the following conditions $(\mathcal{M})^\varepsilon$ and $(\mathcal{K})^\varepsilon$:*

$$(\mathcal{M})^\varepsilon: \quad \varepsilon + \lambda^\varepsilon \int_\Omega \Big(f(t, x^*(t), u(t)) - f(t, x^*(t), u^*(t)) \Big)\, dt - \sum_{i,j} \int_\Omega \big(u_{ij}(t) - u^*_{ij}(t) \big)\, y^\varepsilon_{ij}(t)\, dt \geqslant 0 \qquad \forall\, u \in \mathrm{U}\,;$$

$$(\mathcal{K})^\varepsilon: \quad \lambda^\varepsilon \sum_i \int_\Omega \frac{\partial f}{\partial \xi_i}(t, x^*(t), u^*(t)) \cdot \big(x_i(t) - x^*_i(t) \big)\, dt + \sum_{i,j} \int_\Omega \Big(\frac{\partial x_i}{\partial t_j}(t) - \frac{\partial x^*_i}{\partial t_j}(t) \Big)\, y^\varepsilon_{ij}(t)\, dt = 0 \qquad \forall\, x \in W^{1,p}_0(\Omega, \mathbb{R}^n)\,.$$

The limit passage $\varepsilon \to 0$ within these conditions could be justified only in the special case when the optimal control u^* is piecewise continuous. [22] The parameter ε can also be neglected if the integrability conditions (2.5) are explicitly included in the description of the control domain U. [23] In this case, however, an almost everywhere pointwise reformulation of the maximum condition is impossible. Independently of the previously mentioned authors, some special problems of type $(\mathrm{P})_0$ were treated by CELLINA/PERROTTA and ISHII/LORETI in the context of viscosity solutions of Hamilton-Jacobi equations. [24]

In the next section, we will reformulate and prove Pontryagin's maximum principle for the convex problem $(\mathrm{P})_0$ independently of a parameter $\varepsilon > 0$.

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[19] [CESARI 69], p. 348, Theorem, [RUND 74], p. 128, Theorem 2.1., resp. [KLÖTZLER 76], p. 71, Theorem 2. CESARI's proof requires some integrability conditions for needle variations of the type $\mathbb{1}_{(\Omega \setminus \mathrm{E})}(t) \cdot u^*(t) + \mathbb{1}_{\mathrm{E}}(t) \cdot v_0$ within a sufficiently small neighborhood of u^* ([CESARI 69], p. 347, (β)) while KLÖTZLER has to assume a higher regularity of a solution S of the related Hamilton-Jacobi differential inequality

$$\sum_{j=1}^m \frac{\partial S_j}{\partial t_j}(t,\xi) + \operatorname*{Max}_{v \in \mathrm{V}} H\big(t, \xi, v, \nabla_\xi S(t,\xi), \lambda_0\big) \leqslant 0\,, \tag{2.6}$$

namely $S(t,\xi) \in C^2(\Omega \times \mathbb{R}^n, \mathbb{R}^m)$ in a whole neighborhood of a graph $(t,\xi) = (t, x^*(t))$ where equality holds ([KLÖTZLER 76], p. 70 f.). RUND has to assume the existence of dual variables of class C^2 as well ([RUND 74], p. 127 f.).

[20] [KLÖTZLER 92], [KLÖTZLER/PICKENHAIN 93], [KLÖTZLER/PICKENHAIN 94], [PICKENHAIN 91], [PICKENHAIN 92A], p. 431, Theorem 2, [PICKENHAIN 96], [PICKENHAIN/WAGNER 00A], [PICKENHAIN/WAGNER 00B], [PICKENHAIN/WAGNER 01], [PICKENHAIN/WAGNER 05], [PICKENHAIN/WAGNER 06], [WAGNER 96] and [WAGNER 99], p. 171, Theorem 2.3.

[21] [PICKENHAIN/WAGNER 05], p. 151, Theorem 2.1., with $f_1 \equiv \mathfrak{o}$ and $\psi = x - x^*$.

[22] Ibid., p. 146 f., Theorem 1.1.

[23] [PICKENHAIN 92A], p. 426, Theorem 1, and [WAGNER 99], p. 178, Theorem 3.3.

[24] [CELLINA/PERROTTA 98], [ISHII/LORETI 03A] and [ISHII/LORETI 03B].

b) The maximum principle without the artificial parameter ε.

A thorough revision of the proof of Theorem 2.1. reveals that the introduction of the parameter $\varepsilon > 0$ can be ruled out under otherwise identical assumptions.

Theorem 2.2. (Pontryagin's maximum principle for $(\mathrm{P})_0$ in the smooth-convex case) *Consider the problem $(\mathrm{P})_0$ under the assumptions about the data mentioned in Section 2.a) (in particular, f is only measurable and essentially bounded with respect to t). In addition, assume that $f(t, \xi, v)$ is convex with respect to v for all fixed $(t, \xi) \in \Omega \times \mathbb{R}^n$. If (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$ then there exist multipliers $\lambda_0 > 0$ and $y \in L^q(\Omega, \mathbb{R}^{nm})$, $p^{-1} + q^{-1} = 1$, satisfying the following conditions $(\mathcal{M})$ and $(\mathcal{K})$:*

$$(\mathcal{M}): \quad \lambda_0 \int_\Omega \Big(f(t, x^*(t), u(t)) - f(t, x^*(t), u^*(t)) \Big)\, dt \; - \; \sum_{i,j} \int_\Omega \big(u_{ij}(t) - u^*_{ij}(t) \big)\, y_{ij}(t)\, dt \; \geqslant \; 0 \qquad \forall\, u \in \mathrm{U}\,;$$

$$(\mathcal{K}): \quad \lambda_0 \sum_i \int_\Omega \frac{\partial f}{\partial \xi_i}(t, x^*(t), u^*(t)) \cdot \big(x_i(t) - x^*_i(t) \big)\, dt \; + \; \sum_{i,j} \int_\Omega \Big(\frac{\partial x_i}{\partial t_j}(t) - \frac{\partial x^*_i}{\partial t_j}(t) \Big)\, y_{ij}(t)\, dt \; = \; 0 \qquad \forall\, x \in W^{1,p}_0(\Omega, \mathbb{R}^n)\,.$$

Remark. With the same proof arguments, the mistake in the proof of [PICKENHAIN/WAGNER 00A] , p. 231 ff., Theorem 3.4., (47) – (53), can be corrected without additional assumptions on the structure of u^*.

We formulate the maximum condition from Theorem 2.2. as an almost everywhere pointwise condition as well:

Theorem 2.3. (Pointwise maximum condition for $(\mathrm{P})_0$ in the smooth-convex case) *Under the assumptions of Theorem 2.2., the maximum condition in integrated form $(\mathcal{M})$ implies a maximum condition $(\mathcal{MP})$ which holds pointwise almost everywhere:*

$$(\mathcal{MP}): \quad \lambda_0 \big(f(t, x^*(t), v) - f(t, x^*(t), u^*(t)) \big) \; - \; \sum_{i,j} \big(v_{ij} - u^*_{ij}(t) \big)\, y_{ij}(t) \; \geqslant \; 0 \quad \forall\, v \in \mathrm{K} \;\; (\forall)\, t \in \Omega\,.$$

Let us turn now to the proofs of Theorems 2.2. and 2.3.

c) Proofs.

Proof of Theorem 2.2. • Step 1: *The variational lemma for $(\mathrm{P})_0$.* It reads as follows:

Lemma 2.4. (Variational lemma for the smooth-convex problem $(\mathrm{P})_0$) *Consider the problem $(\mathrm{P})_0$ under the assumptions of Theorem 2.2. If (x^*, u^*) is a global minimizer of $(\mathrm{P})_0$ then it holds for any pair $(x, u) \in W^{1,p}_0(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm})$ satisfying (2.1) – (2.3), i.e. for any admissible pair in $(\mathrm{P})_0$:*

$$\sum_{i=1}^n \int_\Omega \frac{\partial f}{\partial \xi_i}(t, x^*(t), u^*(t)) \cdot \big(x_i(t) - x^*_i(t) \big)\, dt \; + \; \sum_{i=1}^n \sum_{j=1}^m \int_\Omega \frac{\partial f}{\partial v_{ij}}(t, x^*(t), u^*(t)) \cdot \big(u_{ij}(t) - u^*_{ij}(t) \big)\, dt \; \geqslant \; 0\,. \tag{2.7}$$

Proof. In consequence of our assumptions about $(\mathrm{P})_0$, we have for all i, j:

$$\frac{\partial f}{\partial \xi_i}(\,\cdot\,, x^*(\,\cdot\,), u^*(\,\cdot\,))\,, \; \frac{\partial f}{\partial v_{ij}}(\,\cdot\,, x^*(\,\cdot\,), u^*(\,\cdot\,)) \; \in \; L^q(\Omega, \mathbb{R}) \,\cap\, L^\infty(\Omega, \mathbb{R})\,, \;\; p^{-1} + q^{-1} = 1\,. \tag{2.8}$$

Thus both the integrals

$$\langle\, F_x(x^*, u^*)\,,\, x - x^* \,\rangle_{L^q\, -\, L^p} \; = \; \sum_{i=1}^n \int_\Omega \frac{\partial f}{\partial \xi_i}(t, x^*(t), u^*(t)) \cdot \big(x_i(t) - x^*_i(t) \big)\, dt \quad \text{and} \tag{2.9}$$

$$\langle\, F_u(x^*, u^*)\,,\, u - u^*\,\rangle_{L^q\,-\,L^p} \;=\; \sum_{i=1}^{n}\sum_{j=1}^{m}\int_\Omega \frac{\partial f}{\partial v_{ij}}(t, x^*(t), u^*(t)) \cdot \big(\, u_{ij}(t) - u^*_{ij}(t)\,\big)\, dt \tag{2.10}$$

are well-defined for any admissible pair $(x,u) \in W_0^{1,p}(\Omega,\mathbb{R}^n) \times L^p(\Omega,\mathbb{R}^{nm})$, and the statement of the Lemma is equivalent to

$$\delta^+ F(x^*,u^*)\,(x - x^*,\, u - u^*) \;=\; \langle\, F_x(x^*,u^*)\,,\, x - x^*\,\rangle_{L^q\,-\,L^p} \;+\; \langle\, F_u(x^*,u^*)\,,\, u - u^*\,\rangle_{L^q\,-\,L^p} \;\geqslant\; 0 \tag{2.11}$$

for any admissible (x,u). ∎

• **Step 2:** *The variational sets* C *and* D *and its properties.* Similarly to [IOFFE/TICHOMIROW 79] , p. 203, we define two sets C, D $\subset \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm})$ by [25]

$$\begin{aligned}
\mathrm{C} &= \left\{ \begin{pmatrix} \tau_0 + \langle\, F_x(x^*,u^*)\,,\, x - x^*\,\rangle_{L^q\,-\,L^p} \\ G(x,u^*) - G(x^*,u^*) \end{pmatrix} \;\middle|\; \begin{matrix} \tau_0 \geqslant 0 \\ x \in W_0^{1,p}(\Omega,\mathbb{R}^n) \end{matrix} \right\} \\
&\qquad\qquad + \left\{ \begin{pmatrix} \langle\, F_u(x^*,u^*)\,,\, u - u^*\,\rangle_{L^q\,-\,L^p} \\ G(x^*,u) - G(x^*,u^*) \end{pmatrix} \;\middle|\; u \in \mathrm{U} \right\}; \\
&= \left\{ \begin{pmatrix} \tau_0 + \langle\, F_x(x^*,u^*)\,,\, x - x^*\,\rangle_{L^q\,-\,L^p} \\ Jx - Jx^* \end{pmatrix} \;\middle|\; \begin{matrix} \tau_0 \geqslant 0 \\ x \in W_0^{1,p}(\Omega,\mathbb{R}^n) \end{matrix} \right\} \\
&\qquad\qquad + \left\{ \begin{pmatrix} \langle\, F_u(x^*,u^*)\,,\, u - u^*\,\rangle_{L^q\,-\,L^p} \\ -u^* + u \end{pmatrix} \;\middle|\; u \in \mathrm{U} \right\};
\end{aligned} \tag{2.12}$$

$$\mathrm{D} = \left\{ \begin{pmatrix} -\varrho \\ \mathfrak{o} \end{pmatrix} \;\middle|\; \varrho\, > 0 \right\}. \tag{2.13}$$

Obviously, C is a convex set with $\binom{0}{\mathfrak{o}} \in \mathrm{C}$. It is also clear that $\mathrm{C} \cap \mathrm{D} = \emptyset$ since by the variational lemma, from $\binom{\gamma}{\mathfrak{o}} \in \mathrm{C}$ it follows $\gamma \geqslant 0$ and $\binom{\gamma}{\mathfrak{o}} \notin \mathrm{D}$. We claim now

Lemma 2.5. *It holds* $\mathrm{cl}\,(\mathrm{C}) \cap \mathrm{D} = \emptyset$ *as well.*

Proof. Consider a sequence $\{\, \binom{\gamma^N}{z^N} \,\}$, C $\to \binom{\gamma}{\mathfrak{o}}$. Then there exist $\tau^N \geqslant 0$, $x^N \in W_0^{1,p}(\Omega,\mathbb{R}^n)$ and $u^N \in \mathrm{U}$ with

$$\gamma^N \;=\; \tau^N + \langle\, F_x(x^*,u^*)\,,\, x^N - x^*\,\rangle_{L^q\,-\,L^p} \;+\; \langle\, F_u(x^*,u^*)\,,\, u^N - u^*\,\rangle_{L^q\,-\,L^p} \;\to^{\mathbb{R}}\; \gamma\,; \tag{2.14}$$

$$z^N \;=\; Jx^N - u^N \;\to^{L^p(\Omega,\mathbb{R}^{nm})}\; \mathfrak{o}\,. \tag{2.15}$$

Since the set $\mathrm{U} \subset L^p(\Omega,\mathbb{R}^{nm})$ is bounded, convex and closed in norm and thus weakly closed, we can choose a subsequence $\{\, u^{N'} \,\}$ of $\{\, u^N \,\}$ converging weakly to $\widetilde{u} \in \mathrm{U}$. The sequence $\{\, Jx^N \,\}$ is bounded in L^p-norm as well, thus, by equivalence of the norms, $\{\, x^N \,\}$ is bounded in $W_0^{1,p}$-norm as well as in L^p-norm. Consequently, the second and third members in γ^N are bounded, and so is the sequence $\{\, \tau^N \,\}$. The $W_0^{1,p}$-norm bounded sequence $\{\, x^{N'} \,\}$ admits a further subsequence $\{\, x^{N''} \,\}$ converging weakly to $\widetilde{x} \in W_0^{1,p}(\Omega,\mathbb{R}^n)$. This subsequence can be chosen in such a way that the Jacobians $Jx^{N''}$ converge weakly to a function $\widetilde{z} \in L^p(\Omega,\mathbb{R}^{nm})$ as well. From the weak continuity of the differentiation operator, we obtain $\widetilde{z} = J\widetilde{x}$, and from the assumed convergence $(Jx^N - u^N) \to \mathfrak{o}$ we get finally $J\widetilde{x} - \widetilde{u} = \mathfrak{o}$. Consequently, $(\widetilde{x},\widetilde{u})$ forms an admissible pair in $(\mathrm{P})_0$, and the variational lemma yields $\gamma \geqslant 0$. ∎

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[25] Keeping the terminology from [IOFFE/TICHOMIROW 79] , we could also write $G(x,u^*) - G(x^*,u^*) = Jx - Jx^* = \langle\, G_x(x^*,u^*)\,,\, x - x^*\,\rangle$.

• **Step 3:** *Separation of* $\operatorname{cl}\operatorname{cone}(\mathrm{C})$ *and* $\operatorname{cl}\operatorname{cone}(\mathrm{D})$. We make use of the following

Theorem 2.6. (Separation theorem for convex cones) [26] *Consider two closed convex cones* A *and* B *within a separable normed space* E. *If* $\mathrm{A} \cap \mathrm{B} = \{\, \mathfrak{o} \,\}$ *and* A *is locally compact then there exists some linear, continuous functional* $f \colon \mathrm{E} \to \mathbb{R}$ *with* $f < 0$ *on* $\mathrm{A} \setminus (\mathrm{A} \cap (-\mathrm{A}))$, $f = 0$ *on* $(\mathrm{A} \cap (-\mathrm{A})) \cup (\mathrm{B} \cap (-\mathrm{B}))$ *and* $f \geqslant 0$ *on* $\mathrm{B} \setminus (\mathrm{B} \cap (-\mathrm{B}))$.

The theorem can be applied to $\mathrm{E} = \mathbb{R} \times L^p(\Omega, \mathbb{R}^{nm})$, equipped with the norm topology, and to the closed convex cones $\mathrm{A} = \operatorname{cl}\operatorname{cone}(\mathrm{D})$ and $\mathrm{B} = \operatorname{cl}\operatorname{cone}(\mathrm{C})$ since $\operatorname{cl}\operatorname{cone}(\mathrm{D}) = \{\, \binom{-\varrho}{\mathfrak{o}} \mid \varrho \geqslant 0 \,\}$ is locally compact, and by Lemma 2.5. it holds that $\operatorname{cl}\operatorname{cone}(\mathrm{C}) \cap \operatorname{cl}\operatorname{cone}(\mathrm{D}) = \{\, \binom{0}{\mathfrak{o}} \,\}$. We see further that $\mathrm{A} \cap (-\mathrm{A}) = \{\, \binom{0}{\mathfrak{o}} \,\}$. Theorem 2.6. guarantees, then, the existence of some nontrivial linear, continuous functional $(\, \lambda_0 \,,\, y \,) \in \mathbb{R} \times L^q(\Omega, \mathbb{R}^{nm})$, $p^{-1} + q^{-1} = 1$, which separates $\operatorname{cl}\operatorname{cone}(\mathrm{C})$ and $\operatorname{cl}\operatorname{cone}(\mathrm{D})$ properly. This means that

$$\begin{aligned} &\lambda_0 \, \gamma_1 \, + \, \langle\, y \,,\, z_1 \,\rangle_{L^q - L^p} \, \geqslant \, \lambda_0 \, \gamma_2 \, + \, \langle\, y \,,\, z_2 \,\rangle_{L^q - L^p} \quad \forall \tbinom{\gamma_1}{z_1} \in \operatorname{cl}\operatorname{cone}(\mathrm{C}) \;\; \forall \tbinom{\gamma_2}{z_2} \in \operatorname{cl}\operatorname{cone}(\mathrm{D}) \quad \Longrightarrow \\ &\lambda_0 \, \gamma_1 \, + \, \langle\, y \,,\, z_1 \,\rangle_{L^q - L^p} \, \geqslant \, \lambda_0 \, \gamma_2 \quad \forall \tbinom{\gamma_1}{z_1} \in \mathrm{C} \;\; \forall \tbinom{\gamma_2}{\mathfrak{o}} \in \mathrm{D} \,. \end{aligned} \tag{2.16}$$

From this variational inequality, the first-order necessary conditions can be derived as follows.

a) Nonnegativity: $\lambda_0 \geqslant 0$. This is the immediate consequence from inserting $\binom{1}{\mathfrak{o}} \in \mathrm{C}$ (generated with $\tau_0 = 1$, $x = x^*$, $u = u^*$) and $\binom{-1}{\mathfrak{o}} \in \mathrm{D}$ into (2.16).

b) Derivation of $(\mathcal{M})$. Insert into the left-hand side of (2.16) elements $\binom{\gamma_1}{z_1} \in \mathrm{C}$ generated with $\tau_0 = 0$, $x = x^*$ and arbitrary $u \in \mathrm{U}$. Together with the convexity of the integrand f in its last variable, we get $(\mathcal{M})$:

$$\begin{aligned} 0 \, \leqslant \, \lambda_0 \, \langle\, F_u(x^*, u^*) \,,\, u - u^* \,\rangle_{L^q - L^p} - \langle\, y \,,\, u - u^* \,\rangle_{L^q - L^p} & \\ \leqslant \, \lambda_0 \, \big(\, F(x^*, u) - F(x^*, u^*) \,\big) - \langle\, y \,,\, u - u^* \,\rangle_{L^q - L^p} \,. & \end{aligned} \tag{2.17}$$

c) Derivation of $(\mathcal{K})$. Insert into the left-hand side of (2.16) some element $\binom{\gamma_1}{z_1} \in \mathrm{C}$ generated with $\tau_0 = 0$, arbitrary $x \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ and $u = u^*$. Then it follows

$$\lambda_0 \, \langle\, F_x(x^*, u^*) \,,\, x - x^* \,\rangle_{L^q - L^p} \, + \, \langle\, y \,,\, Jx - Jx^* \,\rangle_{L^q - L^p} \, \geqslant \, 0 \,. \tag{2.18}$$

At the same time, inserting the element generated with $\tau_0 = 0$, $(2\, x^* - x) \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ and $u = u^*$, we obtain

$$\lambda_0 \, \langle\, F_x(x^*, u^*) \,,\, x^* - x \,\rangle_{L^q - L^p} \, + \, \langle\, y \,,\, Jx^* - Jx \,\rangle_{L^q - L^p} \, \geqslant \, 0 \,. \tag{2.19}$$

Equations (2.18) and (2.19) give together $(\mathcal{K})$.

• **Step 4:** *Occurrence of the regular case.* Let us assume, on the contrary, that $\lambda_0 = 0$. Then $(\mathcal{K})$ reads as

$$\langle\, y \,,\, Jx \,\rangle_{L^q - L^p} \; = \; \langle\, y \,,\, Jx^* \,\rangle_{L^q - L^p} \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n) \,, \tag{2.20}$$

which implies $\langle\, y \,,\, Jx^* \,\rangle = \langle\, y \,,\, u^* \,\rangle_{L^q - L^p} = 0$. Then from the maximum condition $(\mathcal{M})$ it follows that

$$-\, \langle\, y \,,\, u - u^* \,\rangle_{L^q - L^p} \; = \; -\, \langle\, y \,,\, u \,\rangle_{L^q - L^p} \, \geqslant \, 0 \quad \forall\, u \in \mathrm{U} \,. \tag{2.21}$$

Since $\mathfrak{o} \in \operatorname{int}(\mathrm{K})$ by assumption, U contains some $L^\infty(\Omega, \mathbb{R}^{nm})$-norm ball V, and we conclude

$$\langle\, y \,,\, u \,\rangle_{L^q - L^p} = 0 \quad \forall\, u \in \mathrm{U} \cap \mathrm{V} \,. \tag{2.22}$$

[26] [Klee 55], p. 315, Theorem 2.7.

This means, however, that y vanishes on all functions $z \in C_0^\infty(\Omega, \mathbb{R}^{nm}) \cap L^p(\Omega, \mathbb{R}^{nm})$ and thus on the whole space $L^p(\Omega, \mathbb{R}^{nm})$. [27] Consequently, from $\lambda_0 = 0$ it follows that $y = \mathfrak{o}_{L^q(\Omega, \mathbb{R}^{nm})}$, and we have arrived at a contradiction since the separating hyperplane for $\operatorname{cl}\operatorname{cone}(\mathrm{C})$ and $\operatorname{cl}\operatorname{cone}(\mathrm{D})$ was described by a nontrivial functional. We obtain $\lambda_0 > 0$, and the proof is complete. ■

Proof of Theorem 2.3. Since $\mathrm{K} \subset \mathbb{R}^{nm}$ is a convex body with nonempty interior, the countable subset $\mathrm{K} \cap \mathbb{Q}^{nm}$ is dense in K. Consider the null sets of the non-Lebesgue points of u_{ij}^*, y_{ij}, $f(\,\cdot\,, x^*(\,\cdot\,), u^*(\,\cdot\,))$ and $f(\,\cdot\,, x^*(\,\cdot\,), v)$ for $v \in \mathrm{K} \cap \mathbb{Q}^{nm}$. The countable union N of these subsets is still a Lebesgue null set. Since $\Omega \subset \mathbb{R}^m$ is the closure of a bounded Lipschitz domain, $\partial\Omega$ is a null set as well. [28]
We fix $t_0 \in \operatorname{int}(\Omega) \setminus \mathrm{N}$ and choose a point $v_0 \in \mathrm{K} \cap \mathbb{Q}^{nm}$. Then a closed ball $\mathrm{V} = \mathrm{K}(t_0, \varrho)$ with center in t_0 is contained in $\operatorname{int}(\Omega)$, and the function

$$u(t) = \mathbb{1}_{\mathrm{V}}(t) \cdot \frac{\operatorname{Dist}(\,t, \partial\mathrm{V}\,)}{\operatorname{Dist}(t_0, \partial\mathrm{V})} \cdot \big(\, v_0 - u^*(t)\,\big) \,+\, \mathbb{1}_{(\,\Omega\,\setminus\,\mathrm{V}\,)}(t) \cdot u^*(t) \tag{2.23}$$

belongs to U since $v_0 \in \mathrm{K}$ and $u^*(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$. Further, we have $u(t_0) = v_0$, and t_0 is a Lebesgue point of u. Since the set of Lebesgue points is conserved under linear combinations as well as multiplication by continuous functions, t_0 is a Lebesgue point of $u_{ij}\, y_{ij}$, and from $f(t_0, x^*(t_0), u(t_0)) = f(t_0, x^*(t_0), v_0)$ we see that t_0 is a Lebesgue point of $f(\,\cdot\,, x^*(\,\cdot\,), u(\,\cdot\,))$ as well. Thus we are allowed to form the Lebesgue derivative of $(\mathcal{M})$ at the point t_0 after inserting u into the condition.
From a Vitali covering of Ω, [29] choose some decreasing sequence $\{\,\mathrm{E}^N\,\}$ of closed subsets of Ω with $\bigcap_N \mathrm{E}^N = \{\,t_0\,\}$. Together with u and u^*, all functions

$$u^N(t) = \mathbb{1}_{\mathrm{E}^N}(t) \cdot u(t) + \mathbb{1}_{(\,\Omega\,\setminus\,\mathrm{E}^N\,)}(t) \cdot u^*(t) \tag{2.24}$$

are admissible controls, and from $(\mathcal{M})$ it follows:

$$\begin{aligned} &\lambda_0 \int_\Omega \Big(f(t, x^*(t), u^N(t)) - f(t, x^*(t), u^*(t)) \Big)\, dt \,-\, \sum_{i,j} \int_\Omega \big(\, u_{ij}^N(t) - u_{ij}^*(t)\,\big)\, y_{ij}(t)\, dt \qquad (2.25) \\ &\;=\; \lambda_0 \int_{\mathrm{E}^N} \Big(f(t, x^*(t), u(t)) - f(t, x^*(t), u^*(t)) \Big)\, dt \,-\, \sum_{i,j} \int_{\mathrm{E}^N} \big(\, u_{ij}(t) - u_{ij}^*(t)\,\big)\, y_{ij}(t)\, dt \,\geqslant\, 0 \quad \forall\, N \in \mathbb{N}\,. \end{aligned}$$

When taking the mean values of the integrals, we obtain

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{|\,\mathrm{E}^N\,|} \Big(\lambda_0 \int_{\mathrm{E}^N} \Big(f(t, x^*(t), u(t)) - f(t, x^*(t), u^*(t)) \Big)\, dt \,-\, \sum_{i,j} \int_{\mathrm{E}^N} \big(\, u_{ij}(t) - u_{ij}^*(t)\,\big)\, y_{ij}(t)\, dt \Big) \qquad (2.26) \\ &\;=\; \lambda_0 \big(\, f(t_0, x^*(t_0), u(t_0)) - f(t_0, x^*(t_0), u^*(t_0))\,\big) \,-\, \sum_{i,j} \big(\, u_{ij}(t_0) - u_{ij}^*(t_0)\,\big)\, y_{ij}(t_0) \\ &\;=\; \lambda_0 \big(\, f(t_0, x^*(t_0), v_0) - f(t_0, x^*(t_0), u^*(t_0))\,\big) \,-\, \sum_{i,j} \big(\, (v_0)_{ij} - u_{ij}^*(t_0)\,\big)\, y_{ij}(t_0) \,\geqslant\, 0\,. \qquad (2.27) \end{aligned}$$

The inequality (2.27) holds at every fixed $t_0 \in \operatorname{int}(\Omega) \setminus \mathrm{N}$ for arbitrary $v_0 \in \mathrm{K} \cap \mathbb{Q}^{nm}$. Since its left-hand side is a continuous function of v_0, (2.27) may be extended to the whole set K. ■

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27) [Adams 78], p. 31, Theorem 2.19.

28) [Wagner 06], p. 122, Lemma 9.2.

29) Cf. [Dunford/Schwartz 88], p. 212, Definition 2.

3. Helmholtz-Weyl decomposition of the multipliers.

a) Helmholtz-Weyl decomposition of L_p-spaces.

The classical decomposition of a three-dimensional C^1-vector field into its gradient and curl component [30] can be generalized within the frame of $L^p(\Omega, \mathbb{R}^{nm})$ spaces. [31] We provide some sufficient conditions guaranteeing that the above mentioned subspace $\{ z \in L^p(\Omega, \mathbb{R}^{nm}) \mid \exists\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n) \text{ with } z = Jx \}$ possesses a direct complement within $L^p(\Omega, \mathbb{R}^{nm})$.

Definition 3.1. (Subspaces of "gradient" and "curl" fields) *Let $\Omega \subset \mathbb{R}^m$ be the closure of a bounded domain. Then for $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, we define the following subspaces of $L^p(\Omega, \mathbb{R}^{nm})$ (which are closed in norm):*

$$G^p(\Omega, \mathbb{R}^{nm}) = \Big\{ z \in L^p(\Omega, \mathbb{R}^{nm}) \;\Big|\; \exists\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n) \;\text{ with }\; z = Jx \Big\}; \tag{3.1}$$

$$R^p(\Omega, \mathbb{R}^{nm}) = \Big\{ z \in L^p(\Omega, \mathbb{R}^{nm}) \;\Big|\; \sum_{j=1}^{m} \int_\Omega z_{ij}(t)\, \frac{\partial \psi_i}{\partial t_j}(t)\, dt = 0 \;\; \forall\, \psi_i \in W_0^{1,q}(\Omega, \mathbb{R}),\; 1 \leqslant i \leqslant n \Big\}. \tag{3.2}$$

Within the definition of $R^p(\Omega, \mathbb{R}^{nm})$, one may restrict oneself to test functions $\psi_i \in C_0^\infty(\Omega, \mathbb{R})$ as a dense subset of $W_0^{1,q}(\Omega, \mathbb{R})$.

Theorem 3.2. (Weyl decomposition of $L^p(\Omega, \mathbb{R}^{nm})$) *Let $\Omega \subset \mathbb{R}^m$ be the closure of a bounded Lipschitz domain.*

1)[32] *Assume $m = 2$ and $\frac{4}{3} \leqslant p \leqslant 4$. Then every function $z \in L^p(\Omega, \mathbb{R}^{2n})$ admits a unique decomposition $z = z' + z''$ with $z' \in G^p(\Omega, \mathbb{R}^{2n})$ and $z'' \in R^p(\Omega, \mathbb{R}^{2n})$. Consequently, $L^p(\Omega, \mathbb{R}^{2n}) = G^p(\Omega, \mathbb{R}^{2n}) \oplus R^p(\Omega, \mathbb{R}^{2n})$ can be written as the direct sum of its subspaces $G^p(\Omega, \mathbb{R}^{2n})$ and $R^p(\Omega, \mathbb{R}^{2n})$, and the mapping $z \longmapsto z'$ defines a linear, continuous operator.*

2) Assume $m \geqslant 2$. Then every function $z \in L^2(\Omega, \mathbb{R}^{nm})$ admits a unique decomposition $z = z' + z''$ with $z' \in G^2(\Omega, \mathbb{R}^{nm})$ and $z'' \in R^2(\Omega, \mathbb{R}^{nm})$. Consequently, $L^2(\Omega, \mathbb{R}^{nm}) = G^2(\Omega, \mathbb{R}^{nm}) \oplus R^2(\Omega, \mathbb{R}^{nm})$ can be written as the orthogonal sum of its subspaces $G^2(\Omega, \mathbb{R}^{nm})$ and $R^2(\Omega, \mathbb{R}^{nm})$, and the mapping $z \longmapsto z'$ defines a linear, continuous projection operator.

3) Assume $m \geqslant 2$ and $1 < p < \infty$. If $\partial\Omega$ can be represented by a C^1-curve then every function $z \in L^p(\Omega, \mathbb{R}^{nm})$ admits a unique decomposition $z = z' + z''$ with $z' \in G^p(\Omega, \mathbb{R}^{nm})$ and $z'' \in R^p(\Omega, \mathbb{R}^{nm})$. Consequently, $L^p(\Omega, \mathbb{R}^{nm}) = G^p(\Omega, \mathbb{R}^{nm}) \oplus R^p(\Omega, \mathbb{R}^{nm})$ can be written as the direct sum of its subspaces $G^p(\Omega, \mathbb{R}^{nm})$ and $R^p(\Omega, \mathbb{R}^{nm})$, and the mapping $z \longmapsto z'$ defines a linear, continuous operator.

b) Decomposition of the multipliers within the maximum principle.

Applying the Weyl decomposition theorem to the control variables and the multipliers, the optimality conditions from Section 2.b) can be further refined.

Theorem 3.3. (Weyl decomposition of the multipliers from Theorem 2.2.) *Consider the problem* $(\mathrm{P})_0$ *under the assumptions of Theorem 2.2. together with a global minimizer (x^*, u^*).*

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30) [FICHTENHOLZ 92], p. 334 f.

31) In the literature, there are different termini (Helmholtz decomposition, Weyl decomposition resp. Hodge decomposition). We adapt the terminology from [SIMADER/SOHR 92] and [SIMADER/SOHR 96].

32) [MITREA 02], p. 362, Theorem 4.4., (4.28).

1) Assume $m = 2$ and $\frac{4}{3} \leqslant p \leqslant 4$. Then there exist multipliers $\lambda_0 > 0$, $y' \in G^q(\Omega, \mathbb{R}^{2n})$ and $y'' \in R^q(\Omega, \mathbb{R}^{2n})$, $p^{-1} + q^{-1} = 1$, satisfying the conditions $(\mathcal{M})$ and $(\mathcal{K})$ below.

2) Assume $m \geqslant 2$ and $p = 2$. Then there exist multipliers $\lambda_0 > 0$, $y' \in G^2(\Omega, \mathbb{R}^{nm})$ and $y'' \in R^2(\Omega, \mathbb{R}^{nm})$, satisfying the conditions $(\mathcal{M})$ and $(\mathcal{K})$ below.

3) Assume $m \geqslant 2$ and $1 < p < \infty$. If $\partial\Omega$ can be represented by a C^1-curve then there exist multipliers $\lambda_0 > 0$, $y' \in G^q(\Omega, \mathbb{R}^{nm})$ and $y'' \in R^q(\Omega, \mathbb{R}^{nm})$, $p^{-1} + q^{-1} = 1$, satisfying the conditions $(\mathcal{M})$ and $(\mathcal{K})$ below.

In all of the three cases, the conditions $(\mathcal{M})$ and $(\mathcal{K})$ read as follows while u', u'' denote the components of $u \in L^p(\Omega, \mathbb{R}^{nm})$ in the Weyl decomposition:

$$(\mathcal{M}):\quad \lambda_0 \int_\Omega \big(f(t, x^*(t), u'(t) + u''(t)) - f(t, x^*(t), u^*(t)) \big)\, dt - \sum_{i,j} \int_\Omega \big(u'_{ij}(t) - u^*_{ij}(t) \big)\, y'_{ij}(t)\, dt$$

$$- \sum_{i,j} \int_\Omega u''_{ij}(t)\, y''_{ij}(t)\, dt \;\geqslant\; 0 \quad \forall\, u' \in G^p(\Omega, \mathbb{R}^{nm}),\ u'' \in R^p(\Omega, \mathbb{R}^{nm}) :\ u' + u'' \in \mathrm{U}\,;$$

$$(\mathcal{K}):\quad \lambda_0 \sum_i \int_\Omega \frac{\partial f}{\partial \xi_i}(t, x^*(t), u^*(t)) \cdot \big(x_i(t) - x^*_i(t) \big)\, dt + \sum_{i,j} \int_\Omega \Big(\frac{\partial x_i}{\partial t_j}(t) - \frac{\partial x^*_i}{\partial t_j}(t) \Big)\, y'_{ij}(t)\, dt \;=\; 0$$

$$\forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,.$$

4) Under the assumptions of 1), 2) or 3) the following implication is true: If the integrand f does not depend on ξ then $y' = \mathfrak{o}$.

Remark. Since $u'(t_0)$, $u''(t_0) \in \mathrm{K}$ does not necessarily follow from $u(t_0) = u'(t_0) + u''(t_0) \in \mathrm{K}$, the inclusion of the Weyl decomposition into the pointwise formulation of the maximum condition meets a difficulty.

c) Proofs.

Proof of Theorem 3.2. 2) [33] We derive the assertion from [Simader/Sohr 92], p. 5, Theorem 1.4. and Remark 1.5. After these propositions, every function $z \in L^2(\Omega, \mathbb{R}^m)$ admits a unique Helmholtz decomposition $z = v' + v''$ where

$$v' = \nabla x\,,\ \ x \in W^{1,2}(\Omega, \mathbb{R}) \quad \text{and} \quad v'' \in \mathrm{cl}_{\,L^2(\Omega,\mathbb{R}^m)} \Big(\Big\{ z \in C_0^0(\Omega, \mathbb{R}^m) \ \Big|\ \sum_{j=1}^m \frac{\partial z_j}{\partial t_j}(t) \equiv 0 \Big\} \Big)\,. \tag{3.3}$$

After equipping the space $W_0^{1,2}(\Omega, \mathbb{R})$ with the scalar product

$$\langle\, x\,,\, \psi \,\rangle_{W_0^{1,2} - W_0^{1,2}} \;=\; \int_\Omega \nabla x(t)\, \nabla\psi(t)\, dt \;=\; \langle\, \nabla x\,,\, \nabla\psi \,\rangle_{L^2 - L^2}\,, \tag{3.4}$$

let us define the linear, continuous functional $h(\psi) \colon W_0^{1,2}(\Omega, \mathbb{R}) \to \mathbb{R}$ through $h(\psi) = \langle\, \nabla x\,,\, \nabla\psi \,\rangle_{L^2 - L^2}$. By Riesz' theorem, h admits the unique representation

$$h(\psi) \;=\; \langle\, \nabla x\,,\, \nabla\psi \,\rangle_{L^2 - L^2} \;=\; \langle\, \nabla u\,,\, \nabla\psi \,\rangle_{L^2 - L^2} \quad \forall\, \psi \in W_0^{1,2}(\Omega, \mathbb{R}) \tag{3.5}$$

with some function $u \in W_0^{1,2}(\Omega, \mathbb{R})$. Then we have

$$\langle\, \nabla x - \nabla u\,,\, \nabla\psi \,\rangle_{L^2 - L^2} = -\langle\, x - u\,,\, \Delta\psi \,\rangle_{L^2 - L^2} \;=\; 0 \tag{3.6}$$

for all test functions $\psi \in C_0^\infty(\Omega, \mathbb{R})$, and from Weyl's lemma [34] it follows that $(x - u)$ agrees almost everywhere with a harmonic function. This means particularly that $w = \nabla x - \nabla u$ belongs to $L^2(\Omega, \mathbb{R}^m)$

[33] The following proof was communicated to me by C. Simader on 11. 03. 2005.

[34] [Morrey 66], p. 42, Theorem 2.3.1.

and, consequently, to $R^2(\Omega,\mathbb{R}^m)$ as well. Summing up, we get from the (unique) Helmholtz decomposition $z = v' + v''$ the (unique) Weyl decomposition $z = z' + z''$ with $z' = \nabla u$ and $z'' = v'' + (\nabla x - \nabla u)$. From the definition of the subspaces $G^2(\Omega,\mathbb{R}^m)$ and $R^2(\Omega,\mathbb{R}^m)$ we see that the decomposition is orthogonal.

3) In consequence of our smoothness assumption about $\partial\Omega$, we may apply [Simader/Sohr 96], p. 45, Theorem 1.2, together with p. 11 f., (0.23) – (0.25): Every function $z \in L^p(\Omega,\mathbb{R}^m)$ admits a unique decomposition $z = z' + z''$ with $z' = \nabla x \in G^p(\Omega,\mathbb{R}^m)$ and $z'' \in R^p(\Omega,\mathbb{R}^m)$. Within this decomposition, $x \in W_0^{1,p}(\Omega,\mathbb{R})$ is the unique solution of the variational equality

$$\langle\, \nabla\psi\,,\, \nabla x\,\rangle_{L^q\,-\,L^p} = \langle\, \nabla\psi\,,\, z\,\rangle_{L^q\,-\,L^p} \quad \forall\,\psi \in W_0^{1,q}(\Omega,\mathbb{R})\,. \tag{3.7}$$

Further, the estimate

$$\|\, z'\,\|_{L^p(\Omega,\mathbb{R}^m)} + \|\, z''\,\|_{L^p(\Omega,\mathbb{R}^m)} \leqslant C\,\|\, z\,\|_{L^p(\Omega,\mathbb{R}^m)} \tag{3.8}$$

holds where the constant $C > 0$ depends only on p and Ω. ■

Proof of Theorem 3.3. 1) – 3) In all of the three cases, according to Theorem 3.2. the functions from the space $L^p(\Omega,\mathbb{R}^{nm})$ as well as from $L^q(\Omega,\mathbb{R}^{nm})$, $p^{-1} + q^{-1} = 1$, admit a unique Weyl decomposition. Within the conditions $(\mathcal{M})$ and $(\mathcal{K})$ from Theorem 2.2., we decompose the difference of the controls $(u - u^*)$ as well as the multiplier y, and by Definition 3.1. we obtain

$$-\langle\, y' + y''\,,\, (u' - u^*) + u''\,\rangle_{L^q\,-\,L^p} \;=\; -\langle\, y'\,,\, u' - u^*\,\rangle_{L^q\,-\,L^p} \;-\; \langle\, y''\,,\, u''\,\rangle_{L^q\,-\,L^p} \tag{3.9}$$

and

$$\langle\, y' + y''\,,\, Jx - Jx^*\,\rangle_{L^q\,-\,L^p} \;=\; \langle\, y'\,,\, Jx - Jx^*\,\rangle_{L^q\,-\,L^p}\,. \tag{3.10}$$

Then the conditions take on the claimed shape.

4) If additionally f does not depend on ξ then $(\mathcal{K})$ reduces to

$$\langle\, y'\,,\, Jx - Jx^*\,\rangle_{L^q\,-\,L^p} \;=\; 0 \quad \forall\, x \in W_0^{1,p}(\Omega,\mathbb{R}^n)\,, \tag{3.11}$$

and $y' \in G^q(\Omega,\mathbb{R}^{nm})$ belongs at the same time to $R^q(\Omega,\mathbb{R}^{nm})$ which implies $y' = \mathfrak{o}$. ■

4. Application of multidimensional control to the optical flow problem.

a) The concept of the optical flow.

In this section, we describe *greyscale images* (independently of resolution) through (at least) measurable functions I defined on a rectangle $\Omega \subset \mathbb{R}^2$ with values $0 \leqslant I(t) \leqslant 1$ $(\forall)\, t \in \Omega$. Consider now a family $\{\, I(t,\tau)\,\}$ of greyscale images with identical ranges and zero boundary values (assuming that all images are embedded into a constant frame), depending on an additional time parameter $-T < \tau < T$. Then we look for a phase flux $X(t,\tau) = (\, X_1(t,\tau)\,,\, X_2(t,\tau)\,)^{\mathrm{T}} :\ \Omega \times (\,-T\,,\,T\,) \to \mathbb{R}^2$ which conserves the grey-level values in the process of transformation of images. Thus the flux obeys the equation

$$I(t_1,t_2,\tau) \;=\; I(\,t_1 - X_1(t,\tau)\,,\, t_2 - X_2(T,\tau)\,,\, 0\,)\,, \quad -T < \tau < T \tag{4.1}$$

for all $t \in \Omega$. The inital values in $\tau = 0$ are

$$X_1(t,0) \;=\; 0\,,\;\; \mathrm{X}_2(t,0) \;=\; 0 \tag{4.2}$$

for all $t \in \Omega$. If, moreover, the image data and the vector field $X(t,\tau)$ depend continuously differentiable on the time variable, then differentiation of (4.1) by τ leads to the first-order PDE

$$I_\tau(t,\tau) \;=\; -\,I_{t_1}(\,t - X(t,\tau)\,,\,0\,)\cdot(X_1)_\tau(t,\tau) \;-\; I_{t_2}(\,t - X(t,\tau)\,,\,0\,)\cdot(X_2)_\tau(t,\tau)\,, \tag{4.3}$$

from which there results for $\tau = 0$ the so-called *optical flow constraint*

$$I_{t_1}(t,0)\,(X_1)_\tau(t,0) \;+\; I_{t_2}(t,0)\,(X_2)_\tau(t,0) \;+\; I_\tau(t,0) \;=\; 0 \quad \forall\, t \in \Omega\,. \tag{4.4}$$

The vector fields $X(t,\tau)$ and $x(t,\tau) = \big(\,(X_1)_\tau(t,\tau)\,,\,(X_2)_\tau(t,\tau)\,\big)^{\mathrm{T}}$ are called *optical displacement* resp. *optical flow.* [35] Although in most cases the existence of vector fields X and x with the claimed properties cannot be assured a priori either in local or in global sense, the concept of the optical flow is widely accepted and finds numerous applications, e. g. for compression of video image data, automatic retouching of movie sequences during the process of digitalization, motion tracking or even reconstruction of three-dimensional surfaces by estimation of the disparity map for a stereo image pair. [36]

b) Determination of the optical flow by variational methods.

Since the beginning of the 80s, variational methods were proposed for the determination of the optical flow. Since this quantity is not uniquely determined by equation (4.4) ("aperture problem"), the objective within these problems consists of two terms at least. In the first one, the defect in the equation (4.4) is minimized while the second one is a regularization term involving the first derivatives of x. Then the variational problem (in "spatial formulation") reads as follows:

$$(\mathrm{V})_2: \quad F(x) = \int_\Omega \Big(\, I_{t_1}(t,0)\,x_1(t,0) + I_{t_2}(t,0)\,x_2(t,0) + I_\tau(t,0)\,\Big)^2\,dt \tag{4.5}$$

$$+\,\mu \int_\Omega r(t, \nabla x_1(t,0), \nabla x_2(t,0))\,dt \;\longrightarrow\; \inf!\,; \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^2)$$

where $I \in W^{1,\infty}(\,\Omega \times (\,-T\,,\,T\,)\,,\,\mathbb{R}\,)$, $\mu > 0$ and $r \in C^2(\Omega\,\times\,\mathbb{R}^4, \mathbb{R})$. [37] The proposed regularizing terms can be classified under different viewpoints. On the one hand, for sufficiently smooth solutions, the Euler-Lagrange equations admit an interpretation in terms of diffusion processes. Another classification relies upon the comprehension of the partial derivatives of I which is important for the discernment of "moving edges" within the flow. [38] Let us now quote some typical examples.

1) Quadratic regularization. Here the integrands are of the type

$$r(v) = (v_{11})^2 + (v_{12})^2 + (v_{21})^2 + (v_{22})^2 \tag{4.6}$$

(isotropic, flow-driven regularization) [39] or

$$r(v) = \varphi\big(\,I_{t_1}(t,0)^2 + I_{t_2}(t,0)^2\,\big)\cdot\big(\,(v_{11})^2 + (v_{12})^2 + (v_{21})^2 + (v_{22})^2\,\big) \tag{4.7}$$

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[35] Cf. [Aubert/Kornprobst 02] , pp. 182 ff. Concerning criticism of the concept, cf. for example [Florack/Niessen/Nielsen 98] , p. 265 f.

[36] See [Brox/Bruhn/Weickert 06] , [Grossauer 06] , [Hinterberger/Scherzer 01] , [Hinterberger/Scherzer/Schnörr/Weickert 02] , p. 69 f., and [Slesareva/Bruhn/Weickert 05] .

[37] Recently, the optical flow problem has been investigated for $x \in BV(\Omega, \mathbb{R}^2)$ in the literature as well. See, for example, [Aubert/Deriche/Kornprobst 99] , pp. 162 – 174, [Hinterberger/Scherzer/Schnörr/Weickert 02] , pp. 81 ff., and [Kornprobst/Deriche/Aubert 99] , pp. 9 ff.

[38] [Weickert/Brox 02] , pp. 252 – 258, and [Weickert/Schnörr 01] , pp. 247 – 253.

[39] [Horn/Schunck 81] , p. 191, and [Weickert/Brox 02] , p. 258.

with a monotonically decreasing, positive function $\varphi\colon\ [\,0\,,\,\infty\,) \to (\,0\,,\,\infty\,)$ (isotropic, image-driven regularization). [40]

2) Convex, nonquadratic regularization. Some of the proposed terms are

$$r(v) = 2\,\sqrt{1+(v_{11})^2+(v_{12})^2} + 2\,\sqrt{1+(v_{21})^2+(v_{22})^2} - 4\,, \tag{4.8}$$

(anisotropic, flow-driven regularization) [41] or

$$\begin{aligned} r(v) &= \frac{1}{2\,\alpha^2+(I_{t_1})^2+(I_{t_2})^2}\cdot\mathrm{trace}\left[\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix}\begin{pmatrix} \alpha^2+(I_{t_2})^2 & -I_{t_1}I_{t_2} \\ -I_{t_1}I_{t_2} & \alpha^2+(I_{t_1})^2\end{pmatrix}\begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22}\end{pmatrix}\right] \\ &= \frac{1}{2\,\alpha^2+(I_{t_1})^2+(I_{t_2})^2} \\ &\qquad\cdot\Big(\alpha^2\,\big((v_{11})^2+(v_{12})^2+(v_{21})^2+(v_{22})^2\big)\;+\;\big(I_{t_2}\,v_{11}-I_{t_1}\,v_{12}\,\big)^2+\big(I_{t_2}\,v_{21}-I_{t_1}\,v_{22}\,\big)^2\Big)\end{aligned} \tag{4.9}$$

with $I_{t_1} = I_{t_1}(t,0)$, $I_{t_2} = I_{t_2}(t,0)$ and $\alpha > 0$ (anisotropic, image-driven regularization). [42]

c) Determination of the optical flow as multidimensional control problem.

Since the optical flow x is an artificial quantity, there is a large degree of freedom in the choice of its space of origin. In this choice, one can orientate oneself to the requirements that arise with regard to the further processing and evaluation of the optical flow data, which now represents the image sequence. In this context, the addition of control restrictions of the shape $JX(t) \in \mathrm{K}$ resp. $Jx(t) \in \mathrm{K}$ [43] $(\forall)\,t \in \Omega$ to the variational problem $(\mathrm{V})_2$ presents at least three advantages: a) the presence of a control restriction has a regularizing effect of its own, b) the analytical properties of the reference image $I(t,0)$ carry over to the representations $I(t,\tau) \approx I(\,t_1-\tau\,x_1(t,0)\,,\,t_2-\tau\,x_2(t,0)\,,\,0\,)$ (in particular, the Lipschitz continuity of $I(t,0)$ implies Lipschitz continuity of the compositions), [44] and c) the optimal control can be used for simultaneous edge detection within the flow. [45] When imposing these restrictions, $(\mathrm{V})_2$ will be altered into a Dieudonné-Rashevsky type problem of shape $(\mathrm{P})_0$:

$$(\mathrm{P})_2:\quad F(x,u) = \int_\Omega \Big(\,I_{t_1}(t,0)\,x_1(t) + I_{t_2}(t,0)\,x_2(t) + I_\tau(t,0)\,\Big)^2\,dt \tag{4.10}$$

$$+\,\mu\,\int_\Omega r(t,u_{11}(t),u_{12}(t),u_{21}(t),u_{22}(t))\,dt \;\longrightarrow\; \inf!\,;$$

$$(x,u) \in W^{1,p}_0(\Omega,\mathbb{R}^2)\,\times\,L^\infty(\Omega,\mathbb{R}^4)\,; \tag{4.11}$$

$$Jx(t) \;=\; \begin{pmatrix} u_{11}(t) & u_{12}(t) \\ u_{21}(t) & u_{22}(t)\end{pmatrix} \quad (\forall)\,t\in\Omega\,; \tag{4.12}$$

$$u \,\in\, \mathrm{U} \,=\, \big\{\,u \in L^p(\Omega,\mathbb{R}^4)\;\big|\;u_{11}(t)^2+u_{12}(t)^2+u_{21}(t)^2+u_{22}(t)^2 \,\leqslant\, R^2\;\;(\forall)\,t\in\Omega\,\big\}\,. \tag{4.13}$$

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[40] [Weickert/Brox 02], p. 258, and [Weickert/Schnörr 01], p. 248, (15).

[41] [Aubert/Deriche/Kornprobst 99], p. 163.

[42] [Enkelmann 88], p. 154, [Hinterberger/Scherzer/Schnörr/Weickert 02], p. 70 f., and [Weickert/Brox 02], p. 258.

[43] Here and in the following, the Jacobians $Jx(t)$ contain the partial derivatives with respect to the space coordinates t_1 and t_2 only.

[44] Concerning problems of type $(\mathrm{V})_2$, in the literature the overall assumption is that the image data $I(t,\tau)$ are Lipschitz at least, cf. [Aubert/Deriche/Kornprobst 99], p. 165, (3.12), [Hinterberger/Scherzer/Schnörr/Weickert 02], p. 82, [Weickert/Schnörr 01], p. 253, after (56).

[45] Cf. a forthcoming paper together with C. Brune.

Here we assume $1 \leqslant p < \infty$, $\mu > 0$ and $R > 0$. The dependence of the unknowns (x, u) on the parameter $\tau = 0$ is suppressed in notation. As integrand $r \in C^2(\Omega \times \mathbb{R}^4, \mathbb{R})$ within the regularization term, any of the convex functions from above could be chosen. Following the approach of [PICKENHAIN/WAGNER 00A], pp. 222 – 224, the existence of global minimizers in $(\mathrm{P})_2$ can be guaranteed, and the Pontryagin maximum principle in the shape of Theorems 2.2. and 2.3. is available. In consequence of these theorems, the necessary optimality conditions read as follows:

Theorem 4.1. (Pontryagin's maximum principle for the problem $(\mathrm{P})_2$**)** *Consider* $(\mathrm{P})_2$ *with the following assumptions about the data:* $\Omega \subset \mathbb{R}^2$ *is the closure of a bounded Lipschitz domain,* $1 < p < \infty$, $\mu > 0$, $R > 0$*; the function* $I\colon \Omega \times (-T, T) \to \mathbb{R}$ *with* $0 \leqslant I(t,\tau) \leqslant 1\ \forall (t,\tau) \in \Omega \times (-T, T)$ *is continuously differentiable with respect to all its arguments, and the function* $r(t,v)\colon \Omega \times \mathbb{R}^4 \to \mathbb{R}$ *is twice continuously differentiable and convex with respect to* v.
Then $(\mathrm{P})_2$ *satisfies the assumptions of Theorem 2.2. and 2.3., and for a given global minimizer* (x^*, u^*) *of* $(\mathrm{P})_2$ *there exist multipliers* $\lambda_0 > 0$ *and* $y \in L^q(\Omega, \mathbb{R}^4)$, $p^{-1} + q^{-1} = 1$, *satisfying the following conditions* $(\mathcal{M})$, $(\mathcal{K})$ *and* $(\mathcal{MP})$:

$$(\mathcal{M}):\quad \lambda_0\,\mu \int_\Omega \Big(r(t, u(t)) - r(t, u^*(t)) \Big)\, dt - \sum_{i,j=1}^{2} \int_\Omega \big(u_{ij}(t) - u^*_{ij}(t) \big)\, y_{ij}(t)\, dt \geqslant 0 \quad \forall\, u \in \mathrm{U};$$

$$(\mathcal{K}):\quad 2\,\lambda_0 \int_\Omega \Big(\big(I_{t_1}(t,0)\, x_1^*(t) + I_{t_2}(t,0)\, x_2^*(t) + I_\tau(t,0) \big) \cdot \sum_{i=1}^{2} I_{t_i}(t,0) \big(x_i(t) - x_i^*(t) \big) \Big)\, dt$$
$$+ \sum_{i,j=1}^{2} \int_\Omega \Big(\frac{\partial x_i}{\partial t_j}(t) - \frac{\partial x_i^*}{\partial t_j}(t) \Big)\, y_{ij}(t)\, dt = 0 \quad \forall\, x \in W_0^{1,p}(\Omega, \mathbb{R}^2);$$

$$(\mathcal{MP}):\quad -\lambda_0\,\mu\, r(t, u^*(t)) + \sum_{i,j=1}^{2} u^*_{ij}(t)\, y_{ij}(t) = \operatorname*{Max}_{v \in \mathbb{R}^{2\times 2},\, |v| \leqslant R} \Big(-\lambda_0\,\mu\, r(t,v) + \sum_{i,j=1}^{2} v_{ij}\, y_{ij}(t) \Big) \quad (\forall)\, t \in \Omega.$$

Remarks. a) In two recent papers of BORZÌ/ITO/KUNISCH, the reconstruction resp. smoothing of given image data $\widetilde{I}(t,\tau)$ (which are assumed to be degraded) and the determination of the optical flow are merged into a common optimization problem.[46] The authors search at the same time for representations $I(t,\tau)$ and an optical flow $x(t,\tau)$ minimizing an objective of the shape

$$F(I, x) = \int_\Omega \Big(\widetilde{I}(t,\tau_0) - I(t,\tau_0) \Big)^2 dt + \int_\Omega r\big((x_1)_\tau(t,\tau_0),\ \dots,\ (x_2)_{t_2}(t,\tau_0) \big)\, dt \tag{4.14}$$

under the constraint

$$I_{t_1}(t,\tau)\, x_1(t,\tau) + I_{t_2}(t,\tau)\, x_2(t,\tau) + I_\tau(t,\tau) = 0 \quad \forall\, (t,\tau) \in \Omega \times [0, \tau_0],\ I(t,0) = \widetilde{I}(t,0). \tag{4.15}$$

In this setting as well, it is reasonable to add control restrictions on ∇I resp. Jx to the variational problem.

b) In analogy to [AUBERT/KORNPROBST 02], pp. 80 – 83, one may choose nonconvex regularizing terms of Perona-Malik type, e. g. with the integrand

$$r(v) = \frac{(v_{11})^2 + (v_{12})^2}{1 + (v_{11})^2 + (v_{12})^2} + \frac{(v_{21})^2 + (v_{22})^2}{1 + (v_{21})^2 + (v_{22})^2}, \tag{4.16}$$

46) [BORZÌ/ITO/KUNISCH 02A] and [BORZÌ/ITO/KUNISCH 02B]. In the terminology of IOFFE/TICHOMIROW, however, the problems formulated by these authors are Lagrange problems with an equality constraint without control restrictions on ∇I resp. Jx. Cf. [IOFFE/TICHOMIROW 79], p. 97.

in the variational resp. control problems $(\mathrm{V})_2$ and $(\mathrm{P})_2$ as well. [47] Then instead of problem $(\mathrm{P})_2$ one has to consider its quasiconvex relaxation.

d) Variational and optimal control methods for the determination of the optical displacement.

For the determination of the optical displacement, one can formulate a variational problem as well:

$$(\mathrm{V})_3: \quad F(X) = \int_\Omega \Big(I(t_1, t_2, \tau_0) - I(\, t_1 - X_1(t, \tau_0)\,,\, t_2 - X_2(t, \tau_0)\,,\, 0\,) \Big)^2 \, dt \tag{4.17}$$

$$+ \mu \int_\Omega r(t, \nabla X_1(t, \tau_0), \nabla X_2(t, \tau_0))\, dt \longrightarrow \inf!; \quad X \in W_0^{1,p}(\Omega, \mathbb{R}^2)$$

with $\tau_0 \in (\, -T\,,\, T\,)$, I, μ and r as in $(\mathrm{V})_2$. [48] Analogously to $(\mathrm{V})_2$, $(\mathrm{V})_3$ can be replaced by a multidimensional control problem

$$(\mathrm{P})_3: \quad F(X, u) = \int_\Omega \Big(I(t_1, t_2, \tau_0) - I(\, t_1 - X_1(t)\,,\, t_2 - X_2(t)\,,\, 0\,) \Big)^2 \, dt \tag{4.18}$$

$$+ \mu \int_\Omega r(t, u_{11}(t), u_{12}(t), u_{21}(t), u_{22}(t)) \longrightarrow \inf!;$$

$$(X, u) \in W_0^{1,p}(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^4)\,; \tag{4.19}$$

$$JX(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) \\ u_{21}(t) & u_{22}(t) \end{pmatrix} \quad (\forall)\, t \in \Omega\,; \tag{4.20}$$

$$u \in \mathrm{U} = \big\{\, u \in L^p(\Omega, \mathbb{R}^4) \;\big|\; u_{11}(t)^2 + u_{12}(t)^2 + u_{21}(t)^2 + u_{22}(t)^2 \leqslant R^2 \;\; (\forall)\, t \in \Omega \,\big\} \tag{4.21}$$

where the dependence of the unknowns (X, u) on the parameter $\tau = \tau_0$ is suppressed in notation. For $(\mathrm{P})_3$, we get from Theorems 2.2. and 2.3. the Pontryagin maximum principle in the following shape:

Theorem 4.2. (Pontryagin's maximum principle for the problem $(\mathrm{P})_3$**)** *Consider* $(\mathrm{P})_3$ *with the following assumptions about the data:* $\Omega \subset \mathbb{R}^2$ *is the closure of a bounded Lipschitz domain,* $1 < p < \infty$, $-T < \tau_0 < T$, $\mu > 0$, $R > 0$*; the function* $I(\,\cdot\,, \tau_0)$ *is measurable and essentially bounded with* $0 \leqslant I(t, \tau_0) \leqslant 1$ $(\forall)\, t \in \Omega$*, the function* $I(\,\cdot\,, 0): \Omega \to \mathbb{R}$ *is continuously differentiable with* $0 \leqslant I(t, 0) \leqslant 1\ \forall\, t \in \Omega$*. The function* $r: \Omega \times \mathbb{R}^4 \to \mathbb{R}$ *is twice continuously differentiable and convex with respect to* v*.*
Then $(\mathrm{P})_3$ *satisfies all assumptions of Theorem 2.2. and 2.3., and for a given global minimizer* (X^*, u^*) *of* $(\mathrm{P})_3$ *there exist multipliers* $\lambda_0 > 0$ *and* $y \in L^q(\Omega, \mathbb{R}^4)$, $p^{-1} + q^{-1} = 1$*, satisfying the following conditions* $(\mathcal{M})$, $(\mathcal{K})$ *and* $(\mathcal{MP})$*:*

$$(\mathcal{M}): \quad \lambda_0\, \mu \int_\Omega \Big(r(\,t, u(t)\,) - r(\,t, u^*(t)\,) \Big)\, dt \;-\; \sum_{i,j=1}^{2} \int_\Omega \big(u_{ij}(t) - u^*_{ij}(t) \big)\, y_{ij}(t)\, dt \;\geqslant\; 0 \quad \forall\, u \in \mathrm{U}\,;$$

$$(\mathcal{K}): \quad 2\,\lambda_0 \int_\Omega \Big(\big(I(t_1, t_2, \tau_0) - I(\, t_1 - X_1^*(t)\,,\, t_2 - X_2^*(t)\,,\, 0\,) \big)$$

$$\cdot \sum_{i=1}^{2} I_{t_i}(\, t_1 - X_1^*(t)\,,\, t_2 - X_2^*(t)\,,\, 0\,) \cdot \big(X_i(t) - X_i^*(t)\,\big) \Big)\, dt$$

$$+ \sum_{i,j=1}^{2} \int_\Omega \big((X_i)_{t_j}(t) - (X_i^*)_{t_j}(t)\,\big)\, y_{ij}(t)\, dt \;=\; 0 \quad \forall\, X \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,;$$

$$(\mathcal{MP}): \quad -\lambda_0\, \mu\, r(\,t, u^*(t)\,) + \sum_{i,j=1}^{2} u^*_{ij}(t)\, y_{ij}(t) \;=\; \operatorname*{Max}_{v \,\in\, \mathbb{R}^{2\times 2},\, |\, v\,| \,\leqslant\, R} \Big(-\lambda_0\, \mu\, r(t, v) + \sum_{i,j=1}^{2} v_{ij}\, y_{ij}(t) \Big) \quad (\forall)\, t \in \Omega\,.$$

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[47] Cf. [Wagner 06] , p. 114.

[48] See [Alvarez/Weickert/Sánchez 00] , pp. 41 ff., and [Enkelmann 88] , p. 151.

5. Application of multidimensional control to the Shape-from-Shading problem.

a) Shape from Shading and Horn's equation.

We model a piece of earth's surface by a function graph $\{ (t_1, t_2, X(t_1, t_2))^{\mathrm{T}} \in \mathbb{R}^3 \mid (t_1, t_2) \in \Omega \subset \mathbb{R}^2 \}$ where the function $X\colon \Omega \to \mathbb{R}$ is at least Lipschitz continuous. [49] Assume that a light source with constant intensity is situated in the point at infinity. Then we may imagine that the surface will be illuminated by parallel rays whose direction is given by a unit vector $\mathfrak{e} = (\eta_1, \eta_2, \eta_3)^{\mathrm{T}}$. Assume further that an aerial photograph $I(t_1, t_2)$ of the surface is registered in the image plane $\{ (t_1, t_2, T_3)^{\mathrm{T}} \in \mathbb{R}^3 \mid (t_1, t_2) \in \mathbb{R}^2 \}$ with $\sup_{(t_1, t_2) \in \Omega} X(t_1, t_2) < T_3$ as greyscale image while the intensity $I(t_1, t_2)$ is proportional to the amount of the light reflected by the surface element $do(t_1, t_2)$ in direction t_3. Further, we adapt the assumption widely used in the literature that the surface is Lambertian, i. e. the reflexion is completely diffuse. Then the amount of light reflected by $do(t_1, t_2)$ in some direction will depend only on the cosine of the angle between the illuminant direction $\mathfrak{e}$ and the normal vector $\mathfrak{n}(t_1, t_2, X(t_1, t_2))$ of the surface element $do(t_1, t_2)$. [50] Consequently, for almost all $t \in \Omega$, it holds that

$$I(t_1, t_2) = \mathfrak{e}^{\mathrm{T}}\, \mathfrak{n}(t_1, t_2, X(t_1, t_2)) = \frac{1}{\sqrt{1 + X_{t_1}(t_1, t_2)^2 + X_{t_2}(t_1, t_2)^2}} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -X_{t_1}(t_1, t_2) \\ -X_{t_2}(t_1, t_2) \\ 1 \end{pmatrix}. \tag{5.1}$$

(5.1) is called *Horn's equation.* The problem is now the recovering of the surface's shape from the given greyscale image(s). Thus for a given function $I(t_1, t_2)\colon \Omega \to [\,0\,,\,1\,]$, we look for solutions $X\colon \Omega \to \mathbb{R}$ of Horn's equation which are Lipschitz continuous at least.

The following theorem shows that in general, the Dirichlet boundary-value problem for this PDE is ill-posed. [51]

Theorem 5.1. (Dirichlet boundary-value problem for Horn's equation)

1) **(Vertical illumination)** *Given* $\mathfrak{e} = (\,0\,,\,0\,,\,1\,)^{\mathrm{T}}$ *and* $I \in C^0(\Omega, \mathbb{R})$ *with values* $0 < \varepsilon \leqslant I(t) \leqslant \eta_3 = 1$, *then the Dirichlet boundary-value problem*

$$I(t) \cdot \sqrt{1 + X_{t_1}(t)^2 + X_{t_2}(t)^2} - 1 \;=\; 0 \quad (\forall)\, t \in \Omega\,, \quad X \in W_0^{1,\infty}(\Omega, \mathbb{R}) \tag{5.2}$$

possesses uncountably many solutions $X \in W_0^{1,\infty}(\Omega, \mathbb{R})$, *and there exists a convex body* $\mathrm{K} \subset \mathbb{R}^2$ *such that* $\nabla X(t) \in \mathrm{K}$ *for almost all* $t \in \Omega$ *for any solution* X.

2) **(Almost vertical illumination)** *Given a unit vector* $\mathfrak{e} = (\,\eta_1\,,\,\eta_2\,,\,\eta_3\,)^{\mathrm{T}}$ *and* $I \in C^0(\Omega, \mathbb{R})$ *with values* $0 < \eta_2 + \varepsilon \leqslant I(t) \leqslant 1 - \varepsilon \leqslant \eta_3 < 1$, *then the Dirichlet boundary-value problem*

$$I(t) \cdot \sqrt{1 + X_{t_1}(t)^2 + X_{t_2}(t)^2} \;+\; \eta_1\, X_{t_1}(t) \;+\; \eta_2\, X_{t_2}(t) \;-\; \eta_3 \;=\; 0 \quad (\forall)\, t \in \Omega\,, \quad X \in W_0^{1,\infty}(\Omega, \mathbb{R}) \tag{5.3}$$

possesses uncountably many solutions $X \in W_0^{1,\infty}(\Omega, \mathbb{R})$, *and there exists a convex body* $\mathrm{K} \subset \mathbb{R}^2$ *such that* $\nabla X(t) \in \mathrm{K}$ *for almost all* $t \in \Omega$ *for any solution* X.

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[49] We follow [PIECHULLEK 00], pp. 22 ff., and [BARNES/ZHANG 00], p. 127 f.

[50] [PIECHULLEK 00], p. 24: "Although no natural surface shows exact Lambert reflexion, this simple model describes very precisely the reflectance of bright, fine-grained surfaces."

[51] Cf. [BARNES/ZHANG 00], p. 129, Theorem 1.

Remarks. a) The solutions of (5.2) resp. (5.3) are even dense within some subset of $W^{1,\infty}$-functions with respect to the L^∞-norm topology . [52)]

b) LIONS/ROUY/TOURIN provided an example which shows that without further assumptions about $I(t)$, even a C^1-solution of (5.2) is not uniquely determined. [53)]

b) Reconstruction of the surface by variational methods.

1) Variational formulations involving the second partial derivatives of X; single-image methods. As examples, we present the methods of HORN/BROOKS and ZHENG/CHELLAPPA where the Shape from Shading problem is treated in the framework of calculus of variations. In both of them, the partial derivatives X_{t_1} and X_{t_2} are replaced by unknown functions x_1, x_2. In the objective, the defect in Horn's equation is minimized together with a regularizing term involving the first partial derivatives of x_1 and x_2 and thus the second partial derivatives of the unknown surface X. Furthermore, the objective contains a term related to the defect within the integrability condition for x_1 and x_2. In the method of HORN/BROOKS, the variational problem reads as [54)]

$$(\mathrm{V})_4: \quad F(x_1,x_2) = \int_\Omega \Big(I(t)\cdot\sqrt{1+x_1(t)^2+x_2(t)^2} + \sum_{j=1}^{2} \eta_j\, x_j(t) - \eta_3 \Big)^2 dt \qquad (5.4)$$
$$+ \int_\Omega \Big(\frac{\partial x_1}{\partial t_2}(t) - \frac{\partial x_2}{\partial t_1}(t) \Big)^2 dt \; + \; R(x_1,x_2) \;\longrightarrow\; \inf!\,; \quad x \in W_0^{1,p}(\Omega,\mathbb{R}^2)$$

with $2<p<\infty$. The regularization term used in this method is

$$R(x_1,x_2) \;=\; \int_\Omega \sum_{j=1}^{2}\sum_{k=1}^{2} \Big(\frac{\partial}{\partial t_j} \frac{2\,x_k(t)}{\Big(1+x_1(t)^2+x_2(t)^2\Big)^{1/2}} \Big)^2 dt\,. \qquad (5.5)$$

In the method of ZHENG/CHELLAPPA again, instead of X_{t_1} and X_{t_2} two unknown functions x_1, x_2 are considered. The integrability condition is formulated through a comparison with the partial derivatives of a third unknown function x_3: [55)]

$$(\mathrm{V})_5: \quad F(x_1,x_2,x_3) = \int_\Omega \Big(I(t)\cdot\sqrt{1+x_1(t)^2+x_2(t)^2} + \sum_{j=1}^{2} \eta_j\, x_j(t) - \eta_3 \Big)^2 dt \qquad (5.6)$$
$$+\mu \int_\Omega \Big(\Big(x_1(t) - \frac{\partial x_3}{\partial t_1}(t) \Big)^2 + \Big(x_2(t) - \frac{\partial x_3}{\partial t_2}(t) \Big)^2 \Big) dt \; + \; R(x_1,x_2,x_3) \;\longrightarrow\; \inf!\,; \quad x \in W_0^{1,p}(\Omega,\mathbb{R}^3)$$

with $2<p<\infty$. In this method, the regularization term reads as follows:

$$R(x_1,x_2,x_3) \;=\; \int_\Omega \Big(\frac{\partial I}{\partial t_1}(t) \;-\; \frac{-\eta_1\, x_2(t)^2 + \eta_2\, x_1(t)\, x_2(t) - \eta_3\, x_1(t) - \eta_1}{\Big(1+x_1(t)^2+x_2(t)^2\Big)^{3/2}} \cdot \frac{\partial x_1}{\partial t_1}(t) \qquad (5.7)$$
$$-\; \frac{-\eta_2\, x_1(t)^2 + \eta_1\, x_1(t)\, x_2(t) - \eta_3\, x_2(t) - \eta_2}{\Big(1+x_1(t)^2+x_2(t)^2\Big)^{3/2}} \cdot \frac{\partial x_2}{\partial t_1}(t) \Big)^2 dt$$

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52) [DACOROGNA/MARCELLINI 99], p. 44, proof of Theorem 2.3., (2.24), and [BARNES/ZHANG 00], p. 129, Theorem 1.

53) [LIONS/ROUY/TOURIN 93], p. 329, Fig. 2.

54) [HORN/BROOKS 86], pp. 191 ff., as further development of [IKEUCHI/HORN 81], p. 161 f. We formulate this and the following problem within Sobolev spaces.

55) [ZHENG/CHELLAPPA 91], p. 686.

$$+\int_\Omega \Big(\frac{\partial I}{\partial t_2}(t) - \frac{-\eta_1\, x_2(t)^2 + \eta_2\, x_1(t)\, x_2(t) - \eta_3\, x_1(t) - \eta_1}{\Big(1 + x_1(t)^2 + x_2(t)^2\Big)^{3/2}} \cdot \frac{\partial x_1}{\partial t_2}(t) - \frac{-\eta_2\, x_1(t)^2 + \eta_1\, x_1(t)\, x_2(t) - \eta_3\, x_2(t) - \eta_2}{\Big(1 + x_1(t)^2 + x_2(t)^2\Big)^{3/2}} \cdot \frac{\partial x_2}{\partial t_2}(t) \Big)^2 \, dt\,.$$

The variational problems are solved then numerically by discretization schemes for the Euler-Lagrange equations. [56)]

2) Multiple image methods. Even from a pair of stereo images registered in the image plane $\{\, (t_1, t_2, T_3)^{\mathrm{T}} \in \mathbb{R}^3 \mid (t_1, t_2) \in \mathbb{R}^2 \,\}$, the unique reconstruction of a $W^{1,\infty}$-surface is impossible. [57)] If the surface admits Lambertian reflectance then additional information cannot be obtained by a change of the image plane but rather by registration of multiple greyscale images $I_1(s), \ldots, I_k(s)$ of the same surface under different illuminant directions $\mathfrak{e}_1, \ldots, \mathfrak{e}_k$. The multiple image methods presented by Piechullek are based on this fact. [58)]

c) Shape from Shading as multidimensional control problem.

With regard to Theorem 5.1., the restriction $\nabla X(t) \in \mathrm{K}\ (\forall)\, t \in \Omega$ (which can be interpreted as a slope restriction for the unknown surface X as well) has to be included *explicitly* within the variational formulation of the Shape from Shading problem. In this context, we bring up for discussion four possibilities for treating Shape from Shading as a multidimensional control problem.

1) Immediate search for $X \in W^{1,\infty}(\Omega, \mathbb{R})$, single-image method. We arrive at the following Dieudonné-Rashevsky type problem:

$$(\mathrm{P})_4: \quad F(X,u) = \int_\Omega \Big(I(t) \cdot \sqrt{1 + u_1(t)^2 + u_2(t)^2} + \sum_{j=1}^{2} \eta_j\, u_j(t) - \eta_3 \Big)^2 dt \longrightarrow \inf!\,; \tag{5.8}$$

$$(X,u) \in W_0^{1,p}(\Omega, \mathbb{R}) \times L^p(\Omega, \mathbb{R}^2)\,; \tag{5.9}$$

$$\nabla X(t) = u(t) \quad (\forall)\, t \in \Omega\,; \tag{5.10}$$

$$u \in \mathrm{U} = \big\{\, u \in L^p(\Omega, \mathbb{R}^2) \;\big|\; u(t) \in \mathrm{K}\ \ (\forall)\, t \in \Omega \,\big\} \tag{5.11}$$

with $1 < p < \infty$, $I\colon\ \Omega \to \mathbb{R}$ measurable with $0 \leqslant I(t) \leqslant 1\ (\forall)\, t \in \Omega$, a unit vector $\mathfrak{e} = (\, \eta_1\,,\, \eta_2\,,\, \eta_3\,)^{\mathrm{T}}$ and a convex body $\mathrm{K} \subset \mathbb{R}^2$ with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$.
Since the integrand $f(t,v) = \big(I(t) \cdot \sqrt{1 + (v_1)^2 + (v_2)^2} + \eta_1\, v_1 + \eta_2\, v_2 - \eta_3 \big)^2$ is nonconvex, we replace $(\mathrm{P})_4$ by its convex relaxation. First we define for $t \in \Omega$ the sets

$$\mathrm{M}(t) = \big\{\, v \in \mathbb{R}^2 \;\big|\; I(t) \cdot \sqrt{1 + (v_1)^2 + (v_2)^2} + \eta_1\, v_1 + \eta_2\, v_2 \leqslant \eta_3 \,\big\} \tag{5.12}$$

which are ellipses for $I(t) > 0$ (cf. (5.1)) and closed half-spaces for $I(t) = 0$. Thus for all $t \in \Omega$ the sets $\mathrm{M}(t)$ are convex. The function

$$f^c(t,v) = \mathbb{1}_{(\,\mathbb{R}^2 \setminus \mathrm{M}(t)\,)}(v) \cdot f(t,v) \tag{5.13}$$

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56) Cf. the surveys given in [Durou/Falcone/Sagona 04], [Piechullek 00], p. 16 f., as well as [Zhang/Tsai/Cryer/Shah 99], pp. 691 ff.

57) In this case, however, the unique reconstruction of a C^1-surface is possible for appropriate illuminant directions $\mathfrak{e}$. See [Chambolle 94], p. 9, Theorem 2.

58) [Piechullek 00], p. 18 and pp. 27 ff.

admits a representation $f^c(t,v) = \vartheta\big(g(t,v)\big)$ with $\vartheta(s) = 0$ for $s \leqslant 0$ and $\vartheta(s) = s^2$ for $s > 0$ as well as $g(t,v) = \operatorname{Max}\big(0\,,\,\mathbb{1}_{(\mathbb{R}^2\setminus \mathrm{M}(t))}(v)\cdot I(t)\cdot\sqrt{1+(v_1)^2+(v_2)^2} + \eta_1 v_1 + \eta_2 v_2 - \eta_3\big)$ and is, consequently, for fixed $t \in \Omega$ convex as function of v.[59] Obviously, $f^c(t,v)$ is the convex envelope of $f(t,v)$ with respect to v. Now, we can formulate the relaxed problem:

$$(\mathrm{P})_4^c:\quad F^c(X,u) = \int_\Omega \mathbb{1}_{(\mathbb{R}^2\setminus \mathrm{M}(t))}\big(u(t)\big)\cdot\Big(I(t)\cdot\sqrt{1+u_1(t)^2+u_2(t)^2} + \sum_{j=1}^{2}\eta_j\,u_j(t) - \eta_3\Big)^2\,dt \longrightarrow \inf!\,; \tag{5.14}$$

$$(X,u) \in W_0^{1,p}(\Omega,\mathbb{R})\times L^p(\Omega,\mathbb{R}^2)\,; \tag{5.15}$$

$$\nabla X(t) = u(t) \quad (\forall)\,t\in\Omega\,; \tag{5.16}$$

$$u \in \mathrm{U} = \big\{\,u\in L^p(\Omega,\mathbb{R}^2)\;\big|\;u(t)\in\mathrm{K}\;\;(\forall)\,t\in\Omega\,\big\}\,. \tag{5.17}$$

Theorem 5.2. (Relaxation of $(\mathrm{P})_4$) *Consider $(P)_4$ with the following assumptions about the data: $\Omega\subset\mathbb{R}^2$ is the closure of a bounded Lipschitz domain, the function $I\colon\,\Omega\to\mathbb{R}$ is measurable with $0\leqslant I(t)\leqslant 1$ $(\forall)\,t\in\Omega$, $\mathfrak{e} = (\,\eta_1\,,\,\eta_2\,,\,\eta_3\,)^{\mathrm{T}}$ is a unit vector, and $\mathrm{K}\subset\mathbb{R}^2$ is a closed ball centered in the origin. Then problems $(\mathrm{P})_4$ and $(\mathrm{P})_4^c$ possess the same (finite) minimal value, and for $(\mathrm{P})_4^c$ there exists a global minimizer.*

We can include the results of Section 3 into the statement of the Pontryagin maximum principle for $(\mathrm{P})_4^c$. Let us further remark that, by (5.17), the admissible domain of $(\mathrm{P})_4^c$ is a subset of $W_0^{1,\infty}(\Omega,\mathbb{R})\times L^\infty(\Omega,\mathbb{R}^2)$, and $p\in(\,1\,,\,\infty\,)$ may be chosen arbitrarily in the following theorem.

Theorem 5.3. (Pontryagin's maximum principle for the problem $(\mathrm{P})_4^c$) *Consider $(\mathrm{P})_4$ and $(\mathrm{P})_4^c$ under the assumptions of Theorem 5.2. Then for a given global minimizer (x^*,u^*) of $(\mathrm{P})_4^c$ we have the following optimality conditions:*

1) Let $1<p<\infty$. Since $(\mathrm{P})_4^c$ satisfies the assumptions of Theorem 2.2. and 2.3., there exist multipliers $\lambda_0>0$ and $y\in L^q(\Omega,\mathbb{R}^2)$, $p^{-1}+q^{-1}=1$, with

$$\begin{aligned}(\mathcal{M}):\quad &\lambda_0\Bigg(\int_{\{\,t\in\Omega\;|\;u(t)\in\mathrm{M}(t)\,\}}\Big(I(t)\cdot\sqrt{1+u_1(t)^2+u_2(t)^2} + \sum_{j=1}^{2}\eta_j\,u_j(t) - \eta_3\Big)^2\,dt\\ &\quad-\int_{\{\,t\in\Omega\;|\;u^*(t)\in\mathrm{M}(t)\,\}}\Big(I(t)\cdot\sqrt{1+u_1^*(t)^2+u_2^*(t)^2} + \sum_{j=1}^{2}\eta_j\,u_j^*(t) - \eta_3\Big)^2\,dt\\ &\qquad-\sum_{j=1}^{2}\int_\Omega\big(u_j(t)-u_j^*(t)\big)\,y_j(t)\,dt \geqslant 0\quad\forall\,u\in\mathrm{U}\,;\end{aligned}$$

$$(\mathcal{K}):\quad \sum_{j=1}^{2}\int_\Omega\Big(\frac{\partial X}{\partial t_j}(t)-\frac{\partial X^*}{\partial t_j}(t)\Big)\,y_j(t)\,dt = 0\quad\forall\,X\in W_0^{1,p}(\Omega,\mathbb{R})\,;$$

$$\begin{aligned}(\mathcal{MP}):\quad &-\lambda_0\,\Big(I(t)\,\sqrt{1+u_1^*(t)^2+u_2^*(t)^2} + \sum_{j=1}^{2}\eta_j\,u_j^*(t) - \eta_3\Big)^2 + \sum_{j=1}^{2}u_j^*(t)\,y_j(t)\\ &= \operatorname*{Max}_{v\,\in\,\mathrm{K}\,\cap\,\mathrm{M}(t)}\Big(-\lambda_0\,\Big(I(t)\,\sqrt{1+(v_1)^2+(v_2)^2} + \sum_{j=1}^{2}\eta_j\,v_j - \eta_3\Big)^2 + \sum_{j=1}^{2}v_j\,y_j(t)\Big)\\ &\hspace{6cm}(\forall)\,t\in\Omega\text{ with }u^*(t)\in\mathrm{M}(t)\,;\\ 0 &= \operatorname*{Max}_{v\,\in\,\mathrm{K}\,\cap\,\mathrm{M}(t)}\Big(-\lambda_0\,\Big(I(t)\,\sqrt{1+(v_1)^2+(v_2)^2} + \sum_{j=1}^{2}\eta_j\,v_j - \eta_3\Big)^2 + \sum_{j=1}^{2}v_j\,y_j(t)\Big)\\ &\hspace{6cm}(\forall)\,t\in\Omega\text{ with }u^*(t)\notin\mathrm{M}(t)\,.\end{aligned}$$

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[59] [Rockafellar/Wets 98], p. 50, 2.20 (b).

2) Let $\frac{4}{3} \leqslant p \leqslant 4$. Then $(\mathrm{P})_4^c$ *satisfies the assumptions of Theorem 3.3., 1), 2) and 4) as well. Consequently, there is $y' = \mathfrak{o}$ within the Weyl decomposition $y = y' + y''$ of the multiplier from Part 1), and conditions* $(\mathcal{M})$ *and* $(\mathcal{MP})$ *from Part 1) hold with $\lambda_0 > 0$ and $y = y'' \in R^q(\Omega, \mathbb{R}^2)$, $p^{-1} + q^{-1} = 1$, while* $(\mathcal{K})$ *is automatically satisfied.*

2) Immediate search for $X \in W^{1,\infty}(\Omega, \mathbb{R})$, multiple image method. Assume that for the reconstruction of X, e. g. two aerial photographs $I'(t)$, $I''(t)$ with different illuminant directions $\mathfrak{e}' = (\eta_1', \eta_2', \eta_3')^{\mathrm{T}}$ and $\mathfrak{e}'' = (\eta_1'', \eta_2'', \eta_3'')^{\mathrm{T}}$ are available. Then we may search for a common solution (resp. approximate solution) X of both Horn's equations via the following Dieudonné-Rashevsky type problem:

$$(\mathrm{P})_5: \quad F(X,u) = \int_\Omega \Big(\big(I'(t) \cdot \sqrt{1 + u_1(t)^2 + u_2(t)^2} + \sum_{j=1}^2 \eta_j' \, u_j(t) - \eta_3' \big)^2 \tag{5.18}$$

$$+ \big(I''(t) \cdot \sqrt{1 + u_1(t)^2 + u_2(t)^2} + \sum_{j=1}^2 \eta_j'' \, u_j(t) - \eta_3'' \big)^2 \Big) \, dt \longrightarrow \inf!;$$

$$(X,u) \in W^{1,\infty}(\Omega, \mathbb{R}) \times L^\infty(\Omega, \mathbb{R}^2); \tag{5.19}$$

$$\nabla X(t) = u(t) \quad (\forall)\, t \in \Omega; \tag{5.20}$$

$$u \in \mathrm{U} = \big\{ u \in L^p(\Omega, \mathbb{R}^2) \bigm| u(t) \in \mathrm{K} \ (\forall)\, t \in \Omega \big\} \tag{5.21}$$

with $1 < p < \infty$ and a convex body $\mathrm{K} \subset \mathbb{R}^2$ with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. $(\mathrm{P})_5$ can be treated similarly to $(\mathrm{P})_4$; after convex relaxation, the theorems from Sections 2 and 3 are applicable.

3) Multiple image method with balance of different surface representations. Again, let two photographs $I'(t)$, $I''(t)$ of the unknown surface with different illuminant directions $\mathfrak{e}' = (\eta_1', \eta_2', \eta_3')^{\mathrm{T}}$ and $\mathfrak{e}'' = (\eta_1'', \eta_2'', \eta_3'')^{\mathrm{T}}$ be given. Instead of searching for a common solution of the Horn's equations, we could solve both equations separately, which results in two "approximations" X_1, $X_2 \in W_0^{1,\infty}(\Omega, \mathbb{R})$ for the unknown surface. We have to require then that these solutions are sufficiently close neighbors. The coupling between X_1 and X_2 can be realized by a further term within the objective as well as by introduction of state constraints or control restrictions. With an additional integral term, we arrive at the following problem:

$$(\mathrm{P})_6: \quad F(X,u) = \int_\Omega \Big(\big(I'(t) \cdot \sqrt{1 + u_{11}(t)^2 + u_{12}(t)^2} + \sum_{j=1}^2 \eta_j' \, u_{1j}(t) - \eta_3' \big)^2 \tag{5.22}$$

$$+ \big(I''(t) \cdot \sqrt{1 + u_{21}(t)^2 + u_{22}(t)^2} + \sum_{j=1}^2 \eta_j'' \, u_{2j}(t) - \eta_3'' \big)^2 \Big) \, dt + \mu \int_\Omega \Big(X_1(t) - X_2(t) \Big)^2 dt \longrightarrow \inf!;$$

$$(X,u) \in W^{1,\infty}(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^4); \tag{5.23}$$

$$JX(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) \\ u_{21}(t) & u_{22}(t) \end{pmatrix} \quad (\forall)\, t \in \Omega; \tag{5.24}$$

$$u \in \mathrm{U} = \big\{ u \in L^p(\Omega, \mathbb{R}^4) \bigm| u(t) \in \mathrm{K} \ (\forall)\, t \in \Omega \big\} \tag{5.25}$$

with $1 < p < \infty$, $\mu > 0$ and a convex body $\mathrm{K} \subset \mathbb{R}^4$ with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. This nonconvex problem requires quasiconvex instead of convex relaxation, and the shape of the necessary optimality conditions is not yet known.

4) Control formulation of the models of Horn/Brooks *resp.* Zheng/Chellappa *(single-image method).* When accepting the "loose" formulation of the integrability condition within the models $(\mathrm{V})_4$ and $(\mathrm{V})_5$, we obtain problems for two unknown functions x_1, $x_2 \in W^{1,\infty}(\Omega, \mathbb{R})$. The restriction $\nabla X(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$ carries over to $(\mathrm{V})_4$ as a state constraint

$$\big(x_1(t)\,,\, x_2(t) \big)^{\mathrm{T}} \in \mathrm{K} \tag{5.26}$$

and to $(\mathrm{V})_5$ as state constraint and control restriction:

$$\big(\, x_1(t)\,,\, x_2(t)\,\big)^{\mathrm{T}} \in \mathrm{K}\,, \quad \big(\, (x_3)_{t_1}(t)\,,\, (x_3)_{t_2}(t)\,\big)^{\mathrm{T}} \in \mathrm{K}\,. \tag{5.27}$$

Then further restrictions for the gradients of x_1 and x_2 result in restrictions for the curvature for the representation of the unknown surface. For example, from $(\mathrm{V})_4$ we get the state-constrained Dieudonné-Rashevsky type problem

$$(\mathrm{P})_7: \quad F(x,u) = \int_\Omega \Big(I(t)\cdot\sqrt{1+x_1(t)^2+x_2(t)^2} + \sum_{j=1}^{2} \eta_j\, x_j(t) - \eta_3 \Big)^2 dt \tag{5.28}$$

$$+ \int_\Omega \Big(u_{12}(t) - u_{21}(t) \Big)^2 dt + \mu \int_\Omega f(u(t))\, dt \longrightarrow \inf!\,;$$

$$(x,u) \in W_0^{1,p}(\Omega,\mathbb{R}^2) \times L^p(\Omega,\mathbb{R}^4)\,; \tag{5.29}$$

$$Jx(t) \;=\; \begin{pmatrix} u_{11}(t) & u_{12}(t) \\ u_{21}(t) & u_{22}(t) \end{pmatrix} \quad (\forall)\, t \in \Omega\,; \tag{5.30}$$

$$\big(\, x_1(t)\,,\, x_2(t)\,\big)^{\mathrm{T}} \in \mathrm{K} \quad \forall\, t \in \Omega\,; \tag{5.31}$$

$$u \in \mathrm{U} \;=\; \big\{\, u \in L^p(\Omega,\mathbb{R}^4) \;\big|\; u(t) \in \widetilde{\mathrm{K}} \;\; (\forall)\, t \in \Omega \,\big\} \tag{5.32}$$

with $1 < p < \infty$, $\mu > 0$ and convex bodies $\mathrm{K} \subset \mathbb{R}^2$ and $\widetilde{\mathrm{K}} \subset \mathbb{R}^4$ with $\mathfrak{o} \in \operatorname{int}(\mathrm{K})$ and $\mathfrak{o} \in \operatorname{int}(\widetilde{\mathrm{K}})$. Again, we arrive at a problem which requires quasiconvex relaxation and cannot be treated immediately by the theorems from Sections 2 and 3.[60] Let us mention that the state constraint (5.26) can be regularized by means of the introduction of an additional control variable. Let K be, for example, the ball $\mathrm{K}(\mathfrak{o}, R)$. Then the inequality

$$0 \;\leqslant\; R^2 - x_1(t)^2 - x_2(t)^2 \quad \forall\, t \in \Omega \tag{5.33}$$

can be replaced by the mixed restriction

$$-\varepsilon\, \widetilde{u}(t) \;\leqslant\; R^2 - x_1(t)^2 - x_2(t)^2 \quad (\forall)\, t \in \Omega \tag{5.34}$$

with a parameter $\varepsilon > 0$ and an additional control variable $\widetilde{u} \in L^p(\Omega,\mathbb{R})$, $1 < p < \infty$, $0 \leqslant \widetilde{u}(t) \leqslant 1$ $(\forall)\, t \in \Omega$. Often, in the process of numerical solution, a mixed restriction of this type can be treated more advantageously than the original state constraint.[61]

d) Proofs.

Proof of Theorem 5.1. 1) and **2)**: We apply [Dacorogna/Marcellini 99] , p. 35, Theorem 2.3., to the function $f(t,v)\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(t,v) = I(t)\cdot\sqrt{1+v_1^2+v_2^2} + \eta_1\, v_1 + \eta_2\, v_2 - \eta_3\,. \tag{5.35}$$

Together with I, f is continuous in t and v, and for all fixed $t \in \Omega$, f is in v convex. In our terminology, the coercivity condition from [Dacorogna/Marcellini 99] , p. 34, Definition 2.1., reads as follows: There exist some $w \in \mathbb{R}^2$ with

$$f(t, v + h\, w) \;\geqslant\; C_1\, |\, h\,| - C_2 \tag{5.36}$$

[60] In the convex case, however, [Pickenhain/Wagner 00b] , p. 300 f., Theorem 1.1., may be adapted.

[61] Cf. for example [Rösch/Tröltzsch 03] , p. 138 f., and [Tröltzsch 05b] , p. 630 f.

for all $t \in \Omega$ and all $v \in \mathbb{R}^2$ with $|\,v\,| \leqslant R$ where C_1, C_2 are positive constants depending on R only. Choosing $w = (\,0\,,\,1\,)^{\mathrm{T}}$, we obtain for all $v \in \mathbb{R}^2$ with $|\,v\,| \leqslant R$:

$$\begin{aligned} f(t, v + h\,w) \;&=\; I(t)\cdot\sqrt{1 + v_1^2 + (v_2 + h)^2} + \eta_1\, v_1 + \eta_2\,(v_2 + h) - \eta_3 && (5.37)\\ &\geqslant I(t)\cdot\big|\,v_2 + h\,\big| - \eta_1\,\big|\,v_1\,\big| - \eta_2\,\big|\,v_2 + h\,\big| - \eta_3 && (5.38)\\ &\geqslant \Big(\,I(t) - \eta_2\Big)\cdot\Big(\,\big|\,v_2\,\big| + \big|\,h\,\big|\,\Big) - \eta_1\,\big|\,v_1\,\big| - \eta_3 && (5.39)\\ &\geqslant \varepsilon\,\big|\,h\,\big| - \varepsilon\,R - \eta_1\,R - \eta_3\,. && (5.40)\end{aligned}$$

Thus we have found $C_1(R) = \varepsilon$ and $C_2(R) = -\varepsilon\,R - \eta_1\,R - \eta_3$. Since $f(t, \mathfrak{o}) = I(t) - \eta_3 \leqslant 0$ for all $t \in \Omega$, the zero function gives admissible boundary values. The restriction for the gradients follows from [DACOROGNA/MARCELLINI 99] , p. 44 ff. ■

Proof of Theorem 5.2. Theorem 5.2. follows from [EKELAND/TÉMAM 99] , p. 327, Corollary 2.17., together with p. 334, Proposition 3.4., and p. 335 f., Proposition 3.6. ■

Proof of Theorem 5.3. The only assumption of Theorems 2.2., 2.3. and 3.3. we have yet to check is the differentiability of $f^c(t, v)$ with respect to v. Let $t \in \Omega$ be fixed. Then $f^c(t, v)$ is continuously differentiable in $v_0 \in \operatorname{int}(\,\mathrm{M}(t)\,)$ as well as in $v_0 \in \mathbb{R}^2 \setminus \mathrm{M}(t)$. Choose now $v_0 \in \partial\mathrm{M}(t)$; consequently, we have $f^c(t, v_0) = 0$. Assume that $v_0 + h\binom{1}{0} \in \mathbb{R}^2 \setminus \mathrm{M}(t)\ \forall\, h > 0$. Then it holds

$$\lim_{h\to 0+0}\frac{1}{h}\Big(\,f(\,t\,,\,v_0 + h\tbinom{1}{0}\,) - f(t, v_0)\Big) = \frac{\partial}{\partial v_1}\Big(\,I(t)\cdot\sqrt{1 + (v_{0,1})^2 + (v_{0,2})^2} + \sum_{j=1}^{2}\eta_j\,v_{0,j} - \eta_3\Big)^2 = 0\,. \quad (5.41)$$

From the convexity of $\mathrm{M}(t)$ it follows that $v_0 + h\binom{1}{0} \in \mathrm{M}(t)\ \forall\, h < 0$ and

$$\lim_{h\to 0-0}\frac{1}{h}\Big(\,f(\,t\,,\,v_0 + h\tbinom{1}{0}\,) - f(t, v_0)\Big) \;=\; 0\,, \quad (5.42)$$

and the partial derivative $\partial f^c(t, v_0)/\partial v_1 = 0$ exists. Analogously, we get $\partial f^c(t, v_0)/\partial v_2 = 0$. Finally, one may convince oneself of the continuity of $\nabla_v f^c(t, v)$ on $\mathbb{R}^2 \setminus \operatorname{int}(\,\mathrm{M}(t)\,)$. ■

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