

On the lower semicontinuity of functionals involving Lebesgue or improper Riemann integrals in infinite horizon optimal control problems

Abstract. This paper deals with infinite horizon optimal control problems, which are formulated in weighted Sobolev spaces $W_p^{1,n}(\mathbb{R}^+; \nu)$ and weighted L_p -spaces $L_p^r(\mathbb{R}^+; \nu)$. We ask for the consequences of the interpretation of the integral within the objective as a Lebesgue or an improper Riemann integral. In order to justify the use of both types of integrals, various applications of infinite horizon problems are presented. We provide examples showing that lower semicontinuity may fail for objectives involving Lebesgue as well as improper Riemann integrals. Further we prove a lower semicontinuity theorem for an objective with Lebesgue integral under more restrictive growth conditions on the integrand.

Key words: *Optimal control, infinite horizon, weighted Sobolev spaces, lower semicontinuity, Lebesgue integral, improper Riemann integral.*

AMS subject classification: 26 A 42, 46 E 35, 49 J 15, 49 J 45, 90 A 99

1. Introduction.

a) Optimal control with infinite horizon.

In the present paper we investigate infinite horizon optimal control problems. The motivation for their study arises primarily from economics and biology where the infinite time horizon is a very natural phenomenon. Starting with paper of HALKIN [14], the research in the field of optimal control problems with infinite horizon has been increased dramatically. We just mention a few references, namely the textbook [7] and the related chapters in [11] as well as some papers with background in economics [3], [18], continuum mechanics [17], [23] and biology [13]. Throughout the development of the theory, necessary and sufficient optimality conditions as well as existence results were obtained. However, it must be emphasized that, with very few exceptions (e. g. [4] – [6], [8]), no attention was given to the important question which type of integral is appropriate to the problem formulation — the Lebesgue or the improper Riemann integral.

The first aim of our paper is to demonstrate that different integral types can be useful in applications and lead to completely different theoretical results. Further, we point out that, for a correct setting of the problem, the choice of an appropriate state space is essential. Let us mention that the Lagrange multipliers associated with the constraints belong to the dual of the space wherein the constraint set has a nonempty interior.

Our paper is organized as follows. In order to demonstrate how the *Lebesgue and improper Riemann integrals* as well as *weighted Lebesgue and Sobolev spaces* come into consideration, we start in this section with the presentation of some typical applications. Then we provide the general formulation of the infinite horizon problem. In Section 2, we prepare some analytical tools concerning weighted Lebesgue and Sobolev spaces and the Nemytskij operator. In Section 3, we state the main results of the paper.

In the case of the Lebesgue integral, we provide an example where lower semicontinuity is violated. Under additional assumptions we succeed in proving lower semicontinuity of the objective with respect to an appropriate weak topology.

In the case of the improper Riemann integral, we provide two examples. The first one demonstrates that, for the same data, the different interpretations of the integral lead to different feasible sets. In the second one, the weak lower semicontinuity of the objective fails.

Consequently, in cases where semicontinuity of the objective is missing it is impossible to develop an existence theory via Weierstrass' theorem. By this observation, one is led to treat infinite horizon problems by means of duality theory (cf. [15]), as outlined in [19].

b) Application: Optimal economic growth.

Since RAMSEY'S pioneering work, the problem of optimal economic growth is treated with an infinite time horizon. In a recent version, the problem can be formulated as follows (see [7], p. 6 ff.):

$$J(K, Z, C) = \int_0^{\infty} e^{-\rho t} U(C(t)) dt \longrightarrow \text{Max !}; \quad (1.1)$$

$$F(K(t)) = Z(t) + C(t); \quad (1.2)$$

$$\dot{K}(t) = Z(t) - \mu K(t); \quad (1.3)$$

$$K(0) = k_0. \quad (1.4)$$

Here the production function F and the utility function U are given while the investment resp. consumption rates Z and C and the capital stock K are optimization variables. Under certain assumptions on the data it can be shown that there exists a constant capital level \bar{k} such that, “for any nonnegative value of ϱ the optimal trajectory over an infinite time horizon exists and converges toward \bar{k} , and this is true for any initial state k_0 ” ([4], p. 8). From this property it is clear that the function K cannot belong to any usual Sobolev space but to a weighted space as introduced below.

c) Application: Production-inventory model.

This model was presented in [20], pp. 154 ff.:

$$J(I, P) = \int_0^{\infty} e^{-\varrho t} \left(\frac{h}{2} (I(t) - \hat{I}(t))^2 + \frac{c}{2} (P(t) - \hat{P}(t))^2 \right) dt \longrightarrow \text{Min !}; \quad (2.1)$$

$$\dot{I}(t) = P(t) - S(t); \quad I(0) = i_0. \quad (2.2)$$

Here \hat{I} and \hat{P} are given goal levels for inventory and production, S is the given sales rate, h and c are given positive coefficients, and the actual inventory and production rates I and P are optimization variables.

Again, the optimal trajectory of the problem belongs to a weighted Sobolev space. Since the objective in this problem is similar to the norm in the weighted space $W_p^1(\mathbf{R}^+, \tilde{\nu})$ with $\tilde{\nu}(t) = e^{-\varrho t}$ (see Section 2.b) below), this seems to be very natural. We mention that here and in the preceding example, the integrals have to be understood in the Lebesgue sense. In the next example, this is not clear from the outset.

d) Application: Pest control.

Let X and Y describe the population numbers of two interacting species where X is a pest and Y is its natural predator. Then X and Y obey the following dynamics (see [7], pp. 4 ff., cf. also [13]):

$$\dot{X}(t) = X(t) (1 - Y(t)), \quad X(0) = x_0; \quad (3.1)$$

$$\dot{Y}(t) = Y(t) (X(t) - 1), \quad Y(0) = y_0. \quad (3.2)$$

It is well known that this system admits a nontrivial stationary point, namely $\hat{x} = \hat{y} = 1$, while the trajectories cycle around the equilibrium in a kind of conservative motion. Assume now that the population X is harvested with a rate $0 \leq U \leq U_{\max}$

by treatment with some pesticide. Then Y will be influenced with a rate cU too, and the dynamics of the controlled system become

$$\dot{X}(t) = X(t) (1 - Y(t) - U(t)), \quad X(0) = x_0; \quad (4.1)$$

$$\dot{Y}(t) = Y(t) (X(t) - 1 - cU(t)), \quad Y(0) = y_0. \quad (4.2)$$

The cost functional in the time interval $[0, T]$ is then

$$J(X, Y, U) = \int_0^T (X(t) + \alpha U(t)) dt \longrightarrow \text{Min}!,$$

balancing the cost of the nuisance and the cost of its controlling with a constant $\alpha > 0$. However, “there is no natural reason for bounding the time interval on which the system has to be controlled” ([4], p. 5). In the infinite horizon, the Lebesgue integral

$$\int_0^\infty (X(t) + \alpha U(t)) dt \quad (5)$$

becomes infinite for any admissible control U . When normalizing with respect to X , i. e., replacing the integral by

$$\int_0^\infty ((X(t) - \hat{x}) + \alpha U(t)) dt, \quad (6)$$

the integral has to be understood in Riemann sense while the absolute convergence cannot be guaranteed.

e) General formulation of the infinite horizon problems.

As mentioned before, the infinite horizon control problem

$$(P)_\infty: \quad J(x, u) = \int_0^\infty r(t, x(t), u(t)) \tilde{\nu}(t) dt \longrightarrow \text{Min}!; \quad (7.1)$$

$$(x, u) \in W_p^{1,n}(\mathbf{R}^+, \nu) \times L_p^r(\mathbf{R}^+, \nu); \quad (7.2)$$

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a. e. on } \mathbf{R}^+; \quad x(0) = x_0; \quad (7.3)$$

$$u(t) \in U \subset \mathbf{R}^r \quad \text{a. e. on } \mathbf{R}^+ \quad (7.4)$$

is not well-defined since the interpretation of the integral within the objective is ambiguous. In order to make this point precise, we denote the set of pairs (x, u) satisfying (7.2) – (7.4) by \mathcal{B} and formulate the following basic problems:

$$(P)_\infty^L: \quad J_L(x, u) = L \int_0^\infty r(t, x(t), u(t)) \tilde{\nu}(t) dt \longrightarrow \text{Min}!; \quad (8.1)$$

$$(x, u) \in \mathcal{B} \cap \mathcal{B}_L \quad (8.2)$$

where the integral in the objective is understood as Lebesgue integral, and \mathcal{B}_L consists of all processes $(x, u) \in \mathcal{B}$, which make the Lebesgue integral in (8.1) convergent. In the second problem,

$$(P)_{\infty}^R: \quad J_R(x, u) = R\text{-}\int_0^{\infty} r(t, x(t), u(t)) \tilde{\nu}(t) dt \longrightarrow \text{Min} !; \quad (9.1)$$

$$(x, u) \in \mathcal{B} \cap \mathcal{B}_R, \quad (9.2)$$

the integral in the objective is understood as improper Riemann integral, and \mathcal{B}_R consists of all processes $(x, u) \in \mathcal{B}$, which make the improper Riemann integral in (9.1) (at least conditionally) convergent.

The function ν is a density function in the sense explained below. The function $\tilde{\nu}$ is assumed to be nonnegative, but not necessarily a density function. The weighted spaces $W_p^{1,n}(\mathbf{R}^+, \nu)$ and $L_p^r(\mathbf{R}^+, \nu)$ will be defined in Section 2.b) below.

f) Consequences of the distinction between Lebesgue and improper Riemann integrals.

Let us remind that

$$R\text{-}\int_0^{\infty} f(t) dt := \lim_{T \rightarrow \infty} R\text{-}\int_0^T f(t) dt \quad (10)$$

where $f: \mathbf{R}^+ \rightarrow \mathbf{R}$ has to be R-integrable over any closed interval $[0, T] \subset \mathbf{R}^+$. If the Lebesgue integral converges absolutely, i. e.

$$L\text{-}\int_0^{\infty} |f(t)| dt < \infty, \quad (11)$$

then the Lebesgue and the improper Riemann integral coincide,

$$L\text{-}\int_0^{\infty} f(t) dt = R\text{-}\int_0^{\infty} f(t) dt = \lim_{T \rightarrow \infty} L\text{-}\int_0^T f(t) dt. \quad (12)$$

It may happen, however, as in the famous example

$$R\text{-}\int_0^{\infty} \frac{\sin t}{t} dt, \quad (13)$$

that the improper Riemann integral converges conditionally while the Lebesgue integral over the same domain does not exist (see [10], pp. 150 ff.). As a consequence of these facts, the feasible domains \mathcal{B}_L and \mathcal{B}_R are, in general, incomparable. Applying

the Lebesgue integral, we exclude from \mathcal{B} all feasible trajectories which make the improper Riemann integral non-absolutely convergent. On the other hand, taking the improper Riemann integral, all trajectories from \mathcal{B} which are Lebesgue integrable but not Riemann integrable even on compact sets will get lost. For these reasons, it is very important to formulate an infinite horizon problem with the proper integral notion reflecting the situation behind the model in an appropriate way. As we will see in Section 3 below, the problems with distinct integral types will also require completely different mathematical treatments.

2. Some background from calculus and functional analysis.

a) Basic notations.

Let us write $[0, \infty) = \mathbf{R}^+$. We denote by $M^n(\mathbf{R}^+)$, $L_p^n(\mathbf{R}^+)$ and $C^{0,n}(\mathbf{R}^+)$ the spaces of all vector functions $x: \mathbf{R}^+ \rightarrow \mathbf{R}^n$ with Lebesgue measurable, in the p th power Lebesgue integrable or continuous components, respectively (cf. [9], p. 146 and pp. 285 ff., [10], pp. 228 ff.). The Sobolev space $W_p^{1,n}(\mathbf{R}^+)$ is defined then as the space of all vector functions $x: \mathbf{R}^+ \rightarrow \mathbf{R}^n$, whose components belong to $L_p(\mathbf{R}^+)$ and admit distributional derivatives \dot{x}_i (cf. [22], p. 49) belonging to $L_p(\mathbf{R}^+)$ as well. For $n = 1$, we suppress the superscript in the labels of the spaces. The interpretation of the integrals will be made precise by the symbols $L\text{-}\int$ for a Lebesgue and $R\text{-}\int$ for a Riemann integral.

b) Weighted Lebesgue and Sobolev spaces.

A Lebesgue measurable function $\nu: \mathbf{R}^+ \rightarrow \mathbf{R}^+ \setminus \{0\}$ with positive values is called a *density function* iff it is Lebesgue integrable over \mathbf{R}^+ :

$$L\text{-}\int_0^\infty \nu(t) dt < \infty. \quad (14)$$

By means of a density function $\nu \in C^0(\mathbf{R}^+)$, we define for any $1 \leq p < \infty$ the *weighted Lebesgue space*

$$L_p^n(\mathbf{R}^+, \nu) = \left\{ x \in M^n(\mathbf{R}^+) \mid \|x\|_{L_p^n(\mathbf{R}^+, \nu)} = \left(L\text{-}\int_0^\infty |x(t)|^p \nu(t) dt \right)^{1/p} < \infty \right\} \quad (15)$$

as well as

$$L_\infty^n(\mathbf{R}^+, \nu) = \left\{ x \in M^n(\mathbf{R}^+) \mid \|x\|_{L_\infty^n(\mathbf{R}^+, \nu)} = \operatorname{ess\,sup}_{0 \leq t < \infty} |x(t) \nu(t)| < \infty \right\} \quad (16)$$

and the *weighted Sobolev space*

$$W_p^{1,n}(\mathbf{R}^+, \nu) = \{ x \in M^n(\mathbf{R}^+) \mid x \in L_p^n(\mathbf{R}^+, \nu), \dot{x} \in L_p^n(\mathbf{R}^+, \nu) \} \quad (17)$$

(see [16], p. 11 f.). Equipped with the norm

$$\|x\|_{W_p^{1,n}(\mathbf{R}^+, \nu)} = \|x\|_{L_p^n(\mathbf{R}^+, \nu)} + \|\dot{x}\|_{L_p^n(\mathbf{R}^+, \nu)}, \quad (18)$$

$W_p^{1,n}(\mathbf{R}^+, \nu)$ becomes a Banach space (this can be confirmed analogously to [16], p. 19, Theorem 3.6.). Any linear, continuous functional $\varphi: L_p(\mathbf{R}^+, \nu) \rightarrow \mathbf{R}$ can be represented by a function $y \in L_q(\mathbf{R}^+, \nu)$ with $p^{-1} + q^{-1} = 1$ if $1 < p < \infty$ and $q = \infty$ if $p = 1$:

$$\langle \varphi, x \rangle = \int_0^\infty y(t) x(t) \nu(t) dt \quad \forall x \in L_p^1(\mathbf{R}^+, \nu). \quad (19)$$

We can apply [10], p. 287, Theorem 3.2, since the measure generated by the density function ν is σ -finite on \mathbf{R}^+ .

c) Compact imbedding for weighted Sobolev spaces.

Let us recall first an imbedding result for non-weighted Sobolev spaces over unbounded domains. We state it as

Lemma 2.1. ([1], p. 167 f., Example 6.43, together with pp. 170 ff., Theorem 6.47) *Let $1 \leq p < \infty$. Given a positive, nonincreasing, continuously differentiable function $\nu: \mathbf{R}^+ \rightarrow \mathbf{R}^+ \setminus \{0\}$. By means of ν , we define the open set*

$$\Omega_\nu = \{ (t, \xi) \in \mathbf{R}^2 \mid 0 < t, 0 < \xi < \nu(t) \}. \quad (20)$$

Then the imbedding $W_p^1(\Omega_\nu) \rightarrow L_p(\Omega_\nu)$ is compact iff the function ν satisfies the condition

$$\lim_{t \rightarrow \infty} \frac{\nu(t + \varepsilon)}{\nu(t)} = 0 \quad (21)$$

for every fixed $\varepsilon > 0$.

For weighted Sobolev spaces, we mention the following theorem recently proved by ANTOCI [2]:

Theorem 2.2. ([2], p. 63, Theorem 4.3.) *Given a continuous density function $\nu: \mathbf{R}^+ \rightarrow \mathbf{R}^+ \setminus \{0\}$ as defined above. By means of ν , we define the open set*

$$\Omega_\nu = \{ (t, \xi) \in \mathbf{R}^2 \mid 0 < t, 0 < \xi < \nu(t) \}. \quad (22)$$

If the imbedding $W_p^1(\Omega_\nu) \rightarrow L_p(\Omega_\nu)$ is compact then the imbedding $W_p^1(\mathbf{R}^+, \nu) \rightarrow L_p(\mathbf{R}^+, \nu)$ for the weighted spaces is compact as well.

d) Properties of the Nemytskij operator.

For a given Carathéodory function $g(t, \xi): \mathbf{R}^+ \times \mathbf{R}^s \rightarrow \mathbf{R}$ (i. e., $g(\cdot, \xi)$ is Lebesgue measurable for all $\xi \in \mathbf{R}^s$, and $g(t, \cdot)$ is continuous for all $t \in \mathbf{R}^+$), the insertion of a s -vector function $x(t): \mathbf{R}^+ \rightarrow \mathbf{R}^s$ into g is described by the Nemytskij operator N :

$$(Nx)(t) = g(t, x(t)). \quad (23)$$

Note that the following theorem, which has been stated in [21], is particularly valid for Lebesgue spaces on unbounded domains.

Theorem 2.3. ([18], p. 162, Theorem 19.2., together with p. 154 f., Theorem 19.1.) *Let $g(t, \xi): \mathbf{R}^+ \times \mathbf{R}^s \rightarrow \mathbf{R}$ be a Carathéodory function. Then the Nemytskij operator N associated with g by (23) is a bounded and continuous operator between the spaces $L_p^s(\mathbf{R}^+)$ and $L_{p'}(\mathbf{R}^+)$ with $1 \leq p < \infty$ and $1 \leq p' < \infty$ iff g satisfies the growth condition*

$$|g(t, \xi)| \leq A(t) + B \cdot \sum_{i=1}^s |\xi_i|^{p/p'} \quad \forall (t, \xi) \in \mathbf{R}^+ \times \mathbf{R}^s \quad (24)$$

with a function $A \in L_{p'}(\mathbf{R}^+)$ and a constant $B > 0$. If $p' = \infty$ then the condition (24) has to be replaced by

$$|g(t, \xi)| \leq K \quad \forall (t, \xi) \in \mathbf{R}^+ \times \mathbf{R}^s. \quad (25)$$

with $K > 0$.

3. Semicontinuity of functionals with integrals over $[0, \infty)$.

a) A semicontinuity theorem in the case of Lebesgue integrals.

We state now a semicontinuity theorem for the objective (8.1) involving the Lebesgue integral.

Theorem 3.1. *Let $1 < p < \infty$. Consider a nonnegative integrand $r(t, \xi, v): \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^r \rightarrow \mathbf{R}^+$, a function $\tilde{\nu}(t): \mathbf{R}^+ \rightarrow \mathbf{R}^+ \setminus \{0\}$ and a density $\nu(t): \mathbf{R}^+ \rightarrow \mathbf{R}^+ \setminus \{0\}$ under the following assumptions:*

1) *The function $r(t, \xi, v)$ is continuous with respect to t , continuously differentiable with respect to ξ and v , and convex with respect to v .*

2) *With respect to its second and third argument, the integrand r satisfies the following growth condition:*

$$\begin{aligned} & \left| r\left(t, \frac{\xi_1}{\nu(t)^{1/p}}, \dots, \frac{\xi_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\right) \cdot \tilde{\nu}(t) \right| \\ & \leq A_1(t) + B_1 \cdot \sum_{i=1}^n \frac{|\xi_i|^{p/q}}{\nu(t)^{1/q}} + B_1 \cdot \sum_{k=1}^r \frac{|v_k|^{p/q}}{\nu(t)^{1/q}} \quad \forall (t, \xi, v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^r \end{aligned} \quad (26)$$

with a function $A_1 \in L_1(\mathbf{R}^+)$, a constant $B_1 > 0$, and $p^{-1} + q^{-1} = 1$.

3) *With respect to its second and third argument, the gradient $\nabla_v r$ satisfies the following growth condition:*

$$\begin{aligned} & \left| \nabla_v r\left(t, \frac{\xi_1}{\nu(t)^{1/p}}, \dots, \frac{\xi_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\right) \cdot \frac{\tilde{\nu}(t)}{\nu(t)^{1/p}} \right| \\ & \leq A_2(t) + B_2 \cdot \sum_{i=1}^n \frac{|\xi_i|^{p/q}}{\nu(t)^{1/q}} + B_2 \cdot \sum_{k=1}^r \frac{|v_k|^{p/q}}{\nu(t)^{1/q}} \quad \forall (t, \xi, v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^r \end{aligned} \quad (27)$$

with a function $A_2 \in L_q(\mathbf{R}^+)$, a constant $B_2 > 0$ and $p^{-1} + q^{-1} = 1$.

4) *For the density function ν , the condition*

$$\lim_{t \rightarrow \infty} \frac{\nu(t + \varepsilon)}{\nu(t)} = 0 \quad (28)$$

holds for arbitrary $\varepsilon > 0$.

Consider two sequences $\{x^N\}$, $W_p^{1,n}(\mathbf{R}^+, \nu) \rightarrow x_0 \in W_p^{1,n}(\mathbf{R}^+, \nu)$ and $\{u^N\}$, $L_p^r(\mathbf{R}^+, \nu) \rightarrow u_0 \in L_p^r(\mathbf{R}^+, \nu)$. Then the following lower semicontinuity relation

holds:

$$\begin{aligned} J_L(x_0, u_0) &= L \int_0^\infty r(t, x_0(t), u_0(t)) \tilde{\nu}(t) dt \\ &\leq \liminf_{N \rightarrow \infty} L \int_0^\infty r(t, x^N(t), u^N(t)) \tilde{\nu}(t) dt = \liminf_{N \rightarrow \infty} J_L(x^N, u^N). \end{aligned} \quad (29)$$

Proof. • Step 1: *Compactness of the imbedding $W_p^{1,n}(\mathbf{R}^+, \nu) \rightarrow L_p^n(\mathbf{R}^+, \nu)$.* By assumption 4), we can apply Lemma 2.1. in order to ensure the compactness of the imbedding $W_p^{1,n}(\Omega_\nu) \rightarrow L_p^n(\Omega_\nu)$ where $\Omega_\nu \subset \mathbf{R}^2$ is defined by

$$\Omega_\nu = \{ (t, \xi) \in \mathbf{R}^2 \mid 0 < t, 0 < \xi < \nu(t) \}. \quad (30)$$

Then from Theorem 2.2. we get the compactness of the imbedding $W_p^{1,n}(\mathbf{R}^+, \nu) \rightarrow L_p^n(\mathbf{R}^+, \nu)$. Consequently, from the weak convergence of the sequence $\{x^N\}$ in the space $W_p^{1,n}(\mathbf{R}^+, \nu)$ it follows its convergence in $L_p^n(\mathbf{R}^+, \nu)$ -norm.

• Step 2: *A lower estimate for $J_L(x^N, u^N)$.* From differentiability and convexity of r with respect to v , we derive

$$r(t, x^N(t), u^N(t)) \geq r(t, x^N(t), u_0(t)) \quad (31)$$

$$+ \nabla_v r(t, x^N(t), u_0(t))^T (u^N(t) - u_0(t)) \implies$$

$$r(t, x^N(t), u^N(t)) \tilde{\nu}(t) \geq r(t, x^N(t), u_0(t)) \tilde{\nu}(t) \quad (32)$$

$$+ \nabla_v r(t, x_0(t), u_0(t))^T (u^N(t) - u_0(t)) \tilde{\nu}(t)$$

$$+ (\nabla_v r(t, x^N(t), u_0(t)) - \nabla_v r(t, x_0(t), u_0(t)))^T (u^N(t) - u_0(t)) \tilde{\nu}(t) \implies$$

$$L \int_0^\infty r(t, x^N(t), u^N(t)) \tilde{\nu}(t) dt \quad (33)$$

$$\geq J_1(x^N, u_0) + J_2(x_0, u^N, u_0) + J_3(x^N, x_0, u^N, u_0)$$

with

$$J_1(x^N, u_0) = L \int_0^\infty r(t, x^N(t), u_0(t)) \tilde{\nu}(t) dt; \quad (34.1)$$

$$J_2(x_0, u^N, u_0) = L \int_0^\infty \nabla_v r(t, x_0(t), u_0(t))^T (u^N(t) - u_0(t)) \tilde{\nu}(t) dt; \quad (34.2)$$

$$\begin{aligned} J_3(x^N, x_0, u^N, u_0) &= L \int_0^\infty (\nabla_v r(t, x^N(t), u_0(t)) \\ &\quad - \nabla_v r(t, x_0(t), u_0(t)))^T \cdot (u^N(t) - u_0(t)) \tilde{\nu}(t) dt. \end{aligned} \quad (34.3)$$

The existence of the integrals $J_1(x^N, u_0)$, $J_2(x_0, u^N, u_0)$ and $J_3(x^N, x_0, u^N, u_0)$ on the right-hand side of (33) will be confirmed in Step 4 below.

• **Step 3:** *Consequences of the growth conditions.*

Lemma 3.2. *The Nemytskij operator*

$$(N(x^N, u_0))(t) = r(t, x^N(t), u_0(t)) \quad (35)$$

is a continuous map between the spaces $L_p^{n+r}(\mathbf{R}^+, \nu)$ and $L_1(\mathbf{R}^+, \tilde{\nu})$.

Proof. Since

$$(x^N, u_0) \in L_p^{n+r}(\mathbf{R}^+, \nu) \iff (x^N \nu^{1/p}, u_0 \nu^{1/p}) \in L_p^{n+r}(\mathbf{R}^+) \quad (36)$$

and

$$r(\cdot, x^N(\cdot), u_0(\cdot)) \in L_1(\mathbf{R}^+, \tilde{\nu}) \iff r(\cdot, x^N(\cdot), u_0(\cdot)) \cdot \tilde{\nu}(\cdot) \in L_1(\mathbf{R}^+), \quad (37)$$

the growth condition from Theorem 2.3. reads as

$$\begin{aligned} & \left| r\left(t, \frac{\xi_1}{\nu(t)^{1/p}}, \dots, \frac{\xi_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\right) \cdot \tilde{\nu}(t) \right| \\ & \leq A_1(t) + B_1 \cdot \sum_{i=1}^n \frac{|\xi_i|^{p/q}}{\nu(t)^{1/q}} + B_1 \cdot \sum_{k=1}^r \frac{|v_k|^{p/q}}{\nu(t)^{1/q}} \quad \forall (t, \xi, v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^r \end{aligned} \quad (38)$$

with $A_1 \in L_1(\mathbf{R}^+)$ and $B_1 > 0$. Since this condition was assumed, the Nemytskij operator (35) maps $L_p^{n+r}(\mathbf{R}^+, \nu)$ continuously into $L_1(\mathbf{R}^+, \tilde{\nu})$. ■

Lemma 3.3. *The Nemytskij operator*

$$(N(x_0, u_0))(t) = \left| \nabla_v r(t, x_0(t), u_0(t)) \right| \frac{\tilde{\nu}(t)}{\nu(t)} \quad (39)$$

is a (continuous) map between the spaces $L_p^{n+r}(\mathbf{R}^+, \nu)$ and $L_q(\mathbf{R}^+, \nu)$.

Proof. Since

$$(x_0, u_0) \in L_p^{n+r}(\mathbf{R}^+, \nu) \iff (x_0 \nu^{1/p}, u_0 \nu^{1/p}) \in L_p^{n+r}(\mathbf{R}^+) \quad (40)$$

and

$$\begin{aligned} & \left| \nabla_v r(\cdot, x_0(\cdot), u_0(\cdot)) \right| \frac{\tilde{\nu}(\cdot)}{\nu(\cdot)} \in L_q(\mathbf{R}^+, \nu) \\ & \iff \left| \nabla_v r(\cdot, x_0(\cdot), u_0(\cdot)) \right| \frac{\tilde{\nu}(\cdot)}{\nu(\cdot)^{1/p}} \in L_q(\mathbf{R}^+), \end{aligned} \quad (41)$$

the growth condition from Theorem 2.3. reads as

$$\begin{aligned} & \left| \nabla_v r\left(t, \frac{\xi_1}{\nu(t)^{1/p}}, \dots, \frac{\xi_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\right) \cdot \frac{\tilde{\nu}(t)}{\nu(t)^{1/p}} \right| \\ & \leq A_2(t) + B_2 \cdot \sum_{i=1}^n \frac{|\xi_i|^{p/q}}{\nu(t)^{1/q}} + B_2 \cdot \sum_{k=1}^r \frac{|v_k|^{p/q}}{\nu(t)^{1/q}} \quad \forall (t, \xi, v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^r \end{aligned} \quad (42)$$

with $A_2 \in L_q(\mathbf{R}^+)$ and $B_2 > 0$. Since this condition was assumed, the Nemytskij operator (39) maps the space $L_p^{n+n+r}(\mathbf{R}^+, \nu)$ continuously into $L_q(\mathbf{R}^+, \nu)$. ■

Lemma 3.4. *The Nemytskij operator*

$$(N(x^N, x_0, u_0))(t) = \left| \left(\nabla_v r(t, x^N(t), u_0(t)) - \nabla_v r(t, x_0(t), u_0(t)) \right) \cdot \frac{\tilde{\nu}(t)}{\nu(t)} \right| \quad (43)$$

is a continuous map between the spaces $L_p^{n+n+r}(\mathbf{R}^+, \nu)$ and $L_q(\mathbf{R}^+, \nu)$.

Proof. We have (44)

$$(x^N, x_0, u_0) \in L_p^{n+n+r}(\mathbf{R}^+, \nu) \iff (x^N \nu^{1/p}, x_0 \nu^{1/p}, u_0 \nu^{1/p}) \in L_p^{n+n+r}(\mathbf{R}^+)$$

and

$$\begin{aligned} & \left| \left(\nabla_v r(\cdot, x^N(\cdot), u_0(\cdot)) - \nabla_v r(\cdot, x_0(\cdot), u_0(\cdot)) \right) \cdot \frac{\tilde{\nu}(\cdot)}{\nu(\cdot)} \right| \in L_q(\mathbf{R}^+, \nu) \\ & \iff \left| \left(\nabla_v r(\cdot, x^N(\cdot), u_0(\cdot)) - \nabla_v r(\cdot, x_0(\cdot), u_0(\cdot)) \right) \cdot \frac{\tilde{\nu}(\cdot)}{\nu(\cdot)^{1/p}} \right| \in L_q(\mathbf{R}^+). \end{aligned} \quad (45)$$

Then from assumption 3) we derive the following growth condition:

$$\begin{aligned} & \left| \nabla_v r\left(t, \frac{\tilde{\xi}_1}{\nu(t)^{1/p}}, \dots, \frac{\tilde{\xi}_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\right) \right. \\ & \quad \left. - \nabla_v r\left(t, \frac{\xi_1}{\nu(t)^{1/p}}, \dots, \frac{\xi_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\right) \right| \cdot \frac{\tilde{\nu}(t)}{\nu(t)^{1/p}} \\ & \leq \left| \nabla_v r\left(t, \frac{\tilde{\xi}_1}{\nu(t)^{1/p}}, \dots, \frac{\tilde{\xi}_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\right) \right| \cdot \frac{\tilde{\nu}(t)}{\nu(t)^{1/p}} \\ & \quad + \left| \nabla_v r\left(t, \frac{\xi_1}{\nu(t)^{1/p}}, \dots, \frac{\xi_n}{\nu(t)^{1/p}}, \frac{v_1}{\nu(t)^{1/p}}, \dots, \frac{v_r}{\nu(t)^{1/p}}\right) \right| \cdot \frac{\tilde{\nu}(t)}{\nu(t)^{1/p}} \\ & \leq 2A_2(t) + B_2 \cdot \sum_{i=1}^n \frac{|\tilde{\xi}_i|^{p/q}}{\nu(t)^{1/q}} + B_2 \cdot \sum_{i=1}^n \frac{|\xi_i|^{p/q}}{\nu(t)^{1/q}} + 2B_2 \cdot \sum_{k=1}^r \frac{|v_k|^{p/q}}{\nu(t)^{1/q}} \\ & \quad \forall (t, \tilde{\xi}, \xi, v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^r \end{aligned} \quad (46)$$

$$\leq 2A_2(t) + B_2 \cdot \sum_{i=1}^n \frac{|\tilde{\xi}_i|^{p/q}}{\nu(t)^{1/q}} + B_2 \cdot \sum_{i=1}^n \frac{|\xi_i|^{p/q}}{\nu(t)^{1/q}} + 2B_2 \cdot \sum_{k=1}^r \frac{|v_k|^{p/q}}{\nu(t)^{1/q}} \quad (47)$$

$$\forall (t, \tilde{\xi}, \xi, v) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^r$$

where $A_2 \in L_q(\mathbf{R}^+)$ and $2B_2 > 0$. Consequently, the Nemytskij operator (43) maps the space $L_p^{n+n+r}(\mathbf{R}^+, \nu)$ continuously into $L_q(\mathbf{R}^+, \nu)$. ■

• **Step 4:** The integrals $J_1(x^N, u_0)$, $J_2(x_0, u^N, u_0)$ and $J_3(x^N, x_0, u^N, u_0)$. From Lemma 3.2. we conclude that the integrals $J_1(x^N, u_0)$ are finite for all $N \in \mathbf{N}$. Together with Step 1, we can further derive the limit relation

$$\liminf_{N \rightarrow \infty} J_1(x^N, u_0) = \lim_{N \rightarrow \infty} J_1(x^N, u_0) = L \int_0^\infty r(t, x_0(t), u_0(t)) \tilde{\nu}(t) dt, \quad (48)$$

and the last integral is finite as well. Next we estimate $J_2(x_0, u^N, u_0)$ by Hölder's inequality (see [10], p. 222, Theorem 1.5):

$$\begin{aligned} & |J_2(x_0, u^N, u_0)| \\ &= \left| L \int_0^\infty \nabla_v r(t, x_0(t), u_0(t))^T (u^N(t) - u_0(t)) \tilde{\nu}(t) dt \right| \\ &\leq L \int_0^\infty \left| \nabla_v r(t, x_0(t), u_0(t)) \frac{\tilde{\nu}(t)}{\nu(t)} \right| \cdot |u^N(t) - u_0(t)| \nu(t) dt \\ &\leq \left(L \int_0^\infty \left| \nabla_v r(t, x_0(t), u_0(t)) \frac{\tilde{\nu}(t)}{\nu(t)} \right|^q \nu(t) dt \right)^{1/q} \\ &\quad \cdot \left(L \int_0^\infty |u^N(t) - u_0(t)|^p \nu(t) dt \right)^{1/p} \\ &= \left\| \nabla_v r(\cdot, x_0(\cdot), u_0(\cdot)) \frac{\tilde{\nu}(\cdot)}{\nu(\cdot)} \right\|_{L_q(\mathbf{R}^+, \nu)} \cdot \|u^N - u_0\|_{L_p(\mathbf{R}^+, \nu)}. \end{aligned} \quad (49)$$

From Lemma 3.3. it follows that the first norm in (49) is finite, and $J_2(x_0, u^N, u_0)$ can be understood as the application of a linear, continuous functional to the difference $(u^N - u_0) \in L_p^r(\mathbf{R}^+, \nu)$. Then from the weak convergence $\{u^N\} \rightharpoonup u_0$ in $L_p^r(\mathbf{R}^+, \nu)$ it follows that

$$\liminf_{N \rightarrow \infty} J_2(x_0, u^N, u_0) = \lim_{N \rightarrow \infty} J_2(x_0, u^N, u_0) = 0. \quad (50)$$

Again we apply Hölder's inequality in order to estimate $J_3(x^N, x_0, u^N, u_0)$:

$$\begin{aligned} & |J_3(x^N, x_0, u^N, u_0)| \\ &= \left| L \int_0^\infty (\nabla_v r(t, x^N(t), u_0(t)) - \nabla_v r(t, x_0(t), u_0(t)))^T (u^N(t) - u_0(t)) \tilde{\nu}(t) dt \right| \end{aligned}$$

$$\begin{aligned}
&\leq L \int_0^\infty \left| (\nabla_v r(t, x^N(t), u_0(t)) - \nabla_v r(t, x_0(t), u_0(t))) \frac{\tilde{\nu}(t)}{\nu(t)} \right| \\
&\quad \cdot |u^N(t) - u_0(t)| \nu(t) dt \\
&\leq \left(L \int_0^\infty \left| (\nabla_v r(t, x^N(t), u_0(t)) - \nabla_v r(t, x_0(t), u_0(t))) \frac{\tilde{\nu}(t)}{\nu(t)} \right|^q \nu(t) dt \right)^{1/q} \\
&\quad \cdot \left(L \int_0^\infty |u^N(t) - u_0(t)|^p \nu(t) dt \right)^{1/p} \\
&= \left\| (\nabla_v r(\cdot, x^N(\cdot), u_0(\cdot)) - \nabla_v r(\cdot, x_0(\cdot), u_0(\cdot))) \frac{\tilde{\nu}(\cdot)}{\nu(\cdot)} \right\|_{L_q(\mathbf{R}^+, \nu)} \\
&\quad \cdot \|u^N - u_0\|_{L_p^{\tilde{\nu}}(\mathbf{R}^+, \nu)}. \tag{51}
\end{aligned}$$

Since the weakly convergent sequence $\{u^N\}$ is bounded, the second norm difference in (51) is bounded as well, and from Step 1 and Lemma 3.4. it follows that

$$\lim_{N \rightarrow \infty} \left\| (\nabla_v r(\cdot, x^N(\cdot), u_0(\cdot)) - \nabla_v r(\cdot, x_0(\cdot), u_0(\cdot))) \frac{\tilde{\nu}(\cdot)}{\nu(\cdot)} \right\|_{L_q(\mathbf{R}^+, \nu)} = 0. \tag{52}$$

Consequently, we have

$$\liminf_{N \rightarrow \infty} J_3(x^N, x_0, u^N, u_0) = \lim_{N \rightarrow \infty} J_3(x^N, x_0, u^N, u_0) = 0. \tag{53}$$

• **Step 5:** *The lower semicontinuity relation for J_L .* From (33), (48), (50) and (53) we get

$$\begin{aligned}
&\liminf_{N \rightarrow \infty} J_L(x^N, u^N) \\
&\geq \liminf_{N \rightarrow \infty} J_1(x^N, u_0) + \liminf_{N \rightarrow \infty} J_2(x_0, u^N, u_0) + \liminf_{N \rightarrow \infty} J_3(x^N, x_0, u^N, u_0) \\
&= \lim_{N \rightarrow \infty} J_1(x^N, u_0) + \lim_{N \rightarrow \infty} J_2(x_0, u^N, u_0) + \lim_{N \rightarrow \infty} J_3(x^N, x_0, u^N, u_0) \\
&= L \int_0^\infty r(t, x_0(t), u_0(t)) \tilde{\nu}(t) dt = J_L(x_0, u_0), \tag{54}
\end{aligned}$$

and the proof of Theorem 3.1. is complete. ■

b) Counterexamples in the case of improper Riemann integrals.

In the first example we see that the different interpretations of the integral within the objective of an infinite horizon problem lead to different feasible sets. Moreover, the problem has to be formulated in the framework of weighted spaces.

Example 3.5. Let the integrand $r(t, v) : \mathbf{R}^+ \times \mathbf{R} \rightarrow \mathbf{R}$ be given by

$$r(t, v) = \begin{cases} \frac{\sin t}{t} \cdot v & | \ 2k\pi \leq t \leq (2k+1)\pi; \\ 2 \frac{\sin t}{t} \cdot v & | \ (2k+1)\pi \leq t \leq (2k+2)\pi. \end{cases} \quad (55)$$

Consider the “loosely formulated” infinite horizon control problem

$$(P)_\infty : \quad J(x, u) = - \int_0^\infty r(t, u(t)) dt \longrightarrow \text{Min !}; \quad (56.1)$$

$$(x, u) \in W_p^1(\mathbf{R}^+, \nu) \times L_p(\mathbf{R}^+, \nu); \quad (56.2)$$

$$\dot{x}(t) = u(t) \text{ a. e. on } \mathbf{R}^+, \quad x(0) = 0; \quad (56.3)$$

$$u(t) \in U = [\frac{1}{2}, 1] \text{ a. e. on } \mathbf{R}^+. \quad (56.4)$$

As in Section 1.e), denote by \mathcal{B} the set of pairs fulfilling (56.2) – (56.4), by \mathcal{B}_L the subset of pairs $(x, u) \in \mathcal{B}$, which make $L\text{-}\int_0^\infty (-r(t, u(t))) dt$ convergent, and by \mathcal{B}_R the subset of pairs $(x, u) \in \mathcal{B}$, which make $R\text{-}\int_0^\infty (-r(t, u(t))) dt$ convergent. Since the integral is of type (13) it is obvious that

$$\mathcal{B}_L = \emptyset \quad \text{and} \quad \mathcal{B}_R \neq \emptyset. \quad (57)$$

The optimal control in $(P)_\infty^R$ is given through

$$u^*(t) = \begin{cases} 1 & | \ 2k\pi \leq t \leq (2k+1)\pi; \\ \frac{1}{2} & | \ (2k+1)\pi \leq t \leq (2k+2)\pi. \end{cases} \quad (58)$$

Defining the functions $\tilde{\nu}$ and ν by

$$\tilde{\nu}(t) \equiv 1, \quad \nu(t) = e^{-\varrho t} \quad (59)$$

with $\varrho > 0$, we see that

$$x^* \notin W_p^1(\mathbf{R}^+) \quad \text{but} \quad x^* \in W_p^1(\mathbf{R}^+, \nu) \quad (60)$$

for any $1 < p < \infty$ what justifies the choice of a weighted Sobolev space as the state space.

The second example shows that the objective with an improper Riemann integral can fail to be weakly lower semicontinuous.

Example 3.6. Consider the problem

$$(P)_\infty: J(x, u) = - \int_0^\infty \sin(x_1(t)) dt \longrightarrow \text{Min} !; \quad (61.1)$$

$$(x, u) \in W_p^{1,2}(\mathbf{R}^+, \nu) \times (L_p(\mathbf{R}^+, \nu) \cap C^0(\mathbf{R}^+)); \quad (61.2)$$

$$\dot{x}_1(t) = x_2(t) \text{ a. e. on } \mathbf{R}^+, x_1(0) = 0; \quad (61.3)$$

$$\dot{x}_2(t) = u(t) \text{ a. e. on } \mathbf{R}^+, x_2(0) = 0; \quad (61.4)$$

$$u(t) \in U = [0, 1] \text{ a. e. on } \mathbf{R}^+. \quad (61.5)$$

Let the functions $\tilde{\nu}$ and ν be defined as in (59). Again, we denote by \mathcal{B} the set of pairs fulfilling (61.2) – (61.5), by \mathcal{B}_L the subset of pairs $(x, u) \in \mathcal{B}$ which make $L\text{-}\int_0^\infty (-\sin(x_1(t))) dt$ convergent, and by \mathcal{B}_R the subset of pairs $(x, u) \in \mathcal{B}$ which make $R\text{-}\int_0^\infty (-\sin(x_1(t))) dt$ convergent. In this problem, we get $\mathcal{B}_L \neq \emptyset$, $\mathcal{B}_R \neq \emptyset$ since $(x_0, u_0) \equiv ((0), 0) \in \mathcal{B}_L \cap \mathcal{B}_R$. Consider now the sequence

$$u^N(t) = \frac{2}{N} \quad (62)$$

of admissible controls in $(P)_\infty^R$. Then x_1^N and x_2^N belong to the weighted Sobolev space $W_p^1(\mathbf{R}^+, \nu)$ for any $1 < p < \infty$, and

$$\lim_{N \rightarrow \infty} \|x^N - x_0\|_{W_p^{1,2}(\mathbf{R}^+, \nu)} = 0 \text{ as well as} \quad (63.1)$$

$$\lim_{N \rightarrow \infty} \|u^N - u_0\|_{L_p(\mathbf{R}^+, \nu)} = 0. \quad (63.2)$$

However, the semicontinuity of the objective fails along the sequence $\{(x^N, u^N)\} \rightharpoonup (x_0, u_0) \in W_p^{1,2}(\mathbf{R}^+, \nu) \times L_p(\mathbf{R}^+, \nu)$. We calculate (cf. [12], p. 554, Nr. 491, Examples 3 and 4)

$$R\text{-}\int_0^\infty \sin(x_1^N(t)) dt = R\text{-}\int_0^\infty \sin\left(\frac{t^2}{N}\right) dt = \frac{\sqrt{N}}{2} \cdot R\text{-}\int_0^\infty \frac{\sin s}{\sqrt{s}} ds = \frac{\sqrt{N}}{2} \cdot \sqrt{\frac{\pi}{2}}. \quad (64)$$

Consequently, we have (65)

$$J_R(x_0, u_0) = 0 \neq \lim_{N \rightarrow \infty} J_R(x^N, u^N) = \lim_{N \rightarrow \infty} R\text{-}\int_0^\infty \left(-\sin(x_1^N(t))\right) dt = -\infty,$$

and the functional with the improper Riemann integral is not weakly lower semicontinuous.

c) Counterexample in the case of Lebesgue integrals.

This example was considered by HALKIN [14] in order to prove that the adjoint function y of this problem does not satisfy the *natural* transversality condition $\lim_{T \rightarrow \infty} y(T) = 0$, corresponding to the terminal condition that $x(\infty)$ is free.

Example 3.7. Consider the problem

$$(P)_\infty: \quad J(x, u) = L \int_0^\infty (-(1-x(t))u(t)) dt \longrightarrow \text{Min} !; \quad (66.1)$$

$$(x, u) \in W_p^1(\mathbf{R}^+, \nu) \times L_p(\mathbf{R}^+, \nu); \quad (66.2)$$

$$\dot{x}(t) = (1-x(t))u(t) \text{ a. e. on } \mathbf{R}^+, \quad x(0) = 0; \quad (66.3)$$

$$u(t) \in U = [0, 1] \text{ a. e. on } \mathbf{R}^+. \quad (66.4)$$

Let the functions $\tilde{\nu}$ and ν be defined as in (59). Again, we denote by \mathcal{B} the set of pairs fulfilling (66.2) – (66.4), and by \mathcal{B}_L the subset of pairs $(x, u) \in \mathcal{B}$, which make $L \int_0^\infty (1-x(t))u(t) dt$ convergent. Integrating the separated differential equation (66.3) with the initial condition $x(0) = 0$, we obtain $x(t) = 1 - e^{-F(t)}$ with $F(t) = \int_0^t u(s) ds$. We study the following sequence $\{(x^N, u^N)\}$ of feasible processes:

$$u^N(t) = \begin{cases} 0 & | 0 \leq t < N; \\ \frac{1}{N} & | N \leq t < \infty \end{cases} \quad (67)$$

and

$$x^N(t) = \begin{cases} 0 & | 0 \leq t < N; \\ -e^{1-t/N} + 1 & | N \leq t < \infty. \end{cases} \quad (68)$$

We see again that

$$u^N \notin L_p(\mathbf{R}^+) \quad \text{but} \quad u^N \in L_p(\mathbf{R}^+, \nu) \quad (69)$$

and

$$x^N \notin W_p^1(\mathbf{R}^+) \quad \text{but} \quad x^N \in W_p^1(\mathbf{R}^+, \nu) \quad (70)$$

for any $1 \leq p < \infty$, what justifies the choice of a weighted Sobolev space as the state space. Moreover, $\{(x^N, u^N)\}$ converges to $(x_0, u_0) \equiv (0, 0)$ in $W_p^1(\mathbf{R}^+, \nu) \times$

$L_p(\mathbf{R}^+, \nu)$ -norm since

$$\lim_{N \rightarrow \infty} \|x^N\|_{L_p(\mathbf{R}^+, \nu)} = 0; \quad (71.1)$$

$$\lim_{N \rightarrow \infty} \|\dot{x}^N\|_{L_p(\mathbf{R}^+, \nu)} = 0; \quad (71.2)$$

$$\lim_{N \rightarrow \infty} \|u^N\|_{L_p(\mathbf{R}^+, \nu)} = 0 \quad (71.3)$$

for all $1 \leq p < \infty$. Finally, we obtain

$$\begin{aligned} J_L(x^N, u^N) &= L \int_0^\infty (-(1 - x^N(t)) u^N) dt \\ &= L \int_N^\infty (-(1 - x^N(t)) u^N) dt = -1, \end{aligned} \quad (72)$$

but insertion of $(x_0, u_0) \equiv (0, 0)$ gives

$$J_L(x_0, u_0) = L \int_0^\infty 0 dt = 0. \quad (73)$$

Consequently, we arrive at

$$J_L(x_0, u_0) = 0 > \lim_{N \rightarrow \infty} J_L(x^N, u^N) = -1, \quad (74)$$

and the functional with the Lebesgue integral is neither strongly nor weakly lower semicontinuous within the spaces $W_p^1(\mathbf{R}^+, \nu) \times L_p(\mathbf{R}^+, \nu)$, $1 \leq p < \infty$. Let us finally note that in this example the growth conditions (26) and (27) of Theorem 3.1. are not satisfied, and the theorem is not applicable.

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