

Lecture “Measure and Integration” (winter term 2010/11) — Contents

I. Subjects and history of measure theory

1. The measure problem
2. Integration with respect to a measure
3. Measures as functionals on function spaces
4. Remarks about the history of measure theory

II. Definition ranges for set functions

1. Set operations and characteristic functions
2. Algebraically structured set systems
3. The Borel sets and their generation
4. The system of the k -dimensional Lebesgue null sets
5. The Lebesgue sets and their generation
6. The Cantor discontinuum
 - A) The classical Cantor set
 - B) Implications for the cardinality of \mathcal{B}^1 and \mathcal{L}^1
 - C) A Lebesgue set, which is not Borel measurable
 - D) Generalized Cantor constructions

III. Contents and measures

1. Recall of the definitions, examples, basic properties
2. Continuity properties of measures
3. The uniqueness theorem
 - A) Dynkin systems
 - B) The uniqueness theorem for σ -finite measures
4. The extension theorem
 - A) Outer measures
 - B) Carathéodory’s measurability condition
 - C) The extension theorem
5. The Lebesgue measure
 - A) Application of the extension and uniqueness theorem
 - B) Consistency of the extension approach with the former definitions
 - C) Regularity
 - D) Invariance under rigid motions
6. The Banach-Tarski paradox
 - A) Free groups
 - B) A free subgroup of $SO(3)$
 - C) The Hausdorff paradox
 - D) The Banach-Tarski paradox
7. Survey of decomposition theorems for measures
 - A) The relations $\mu \ll \lambda$ and $\mu \perp \lambda$
 - B) Decomposition theorems for measures
 - C) A decomposition theorem for contents

IV. Measurable functions

1. \mathcal{A} - \mathcal{B}^1 -measurable functions
2. \mathcal{A} - $\overline{\mathcal{B}}^1$ -measurable functions

3. Borel- and Lebesgue-measurable functions
 - A) Definition and relation of both
 - B) Lusin's theorem
 - C) Carathéodory functions and the Scorza-Dragoni theorem
4. Approximation of measurable functions by step functions

V. The measure integral

1. Construction of the measure integral
2. Properties of the measure integral
3. Convergence theorems
 - A) The Monotone Convergence theorem
 - B) Fatou's lemma
 - C) The Dominated Convergence theorem
4. Riemann and Lebesgue integral
 - A) Comparison of both approaches
 - B) The Riemann integral in the Lebesgue theory
5. Further properties of the measure integral (survey)
 - A) Indefinite integral
 - B) Lipschitz functions
 - C) Parameter integrals
6. The $L^p(\Omega, \mu)$ -spaces
 - A) The L^p -norms
 - B) The chain of L^p -spaces
 - C) L^p -convergence
 - D) The mean value theorem
 - E) Baire classes

VI. Signed measures

1. Definition and basic properties
2. The Hahn decomposition of Ω
3. The Jordan decomposition of ν
4. Absolute continuity and the Radon-Nikodym theorem
5. Further decompositions of signed measures
6. Banach spaces of signed contents and measures
7. Signed measures as linear, continuous functionals
 - A) Integration with respect to signed set functions
 - B) The space $(C^0(\Omega))^*$
 - C) The space $(L^\infty(\Omega, \mathcal{B}_\Omega^k, \lambda^k))^*$

VII. Applications of measure theory in the context of optimization

1. Identification of generalized derivatives as measures
 - A) Schwartz distributions
 - B) The space $BV(\Omega)$
 - C) Second derivatives of convex functions
2. Measures as Lagrange multipliers related to state constraints
 - A) State-constrained optimization problems in Sobolev spaces
 - B) The subdifferential of convex functions
 - C) The subdifferential for locally Lipschitz functions and the multiplier rule
3. Bochner integration
 - A) Motivation and basic ideas
 - B) Selected spaces of Bochner integrable functions
 - C) Imbedding theorems
 - D) The principal theorem of the calculus in the Bochner version
4. Young measures

VIII. A list of important topics not treated within this course