Priv.-Doz. Dr. Marcus Wagner
University of Graz, Institute for Mathematics and Scientific Computing
Lecture “Measure and Integration” (HS 2010/11)

“Measure and Integration” — Examination Questions (I)

1. What is a content and a measure?
2. State the Banach-Tarski paradox in $\mathbb{R}^3$.
3. Define: k-dimensional Lebesgue measure.
4. Prove that the set of rational numbers forms a 1-dimensional Lebesgue null set within $\mathbb{R}$.
5. For a given set sequence $\{ \mathrm{A}_n \}$, define the sets $\limsup_{n\to\infty} \mathrm{A}_n$ and $\liminf_{n\to\infty} \mathrm{A}_n$ and describe them by characteristic functions.
6. Discuss existence and possible representation of the limit of a monotone set sequence $\{ \mathrm{A}_n \}$.
7. Define: ring, algebra, σ-algebra of sets.
8. What is a generator of a σ-algebra?
9. Explain the definition of the k-dimensional Borel sets $\mathcal{B}^k \subset \mathfrak{P}(\mathbb{R}^k)$.
10. How the k-dimensional Lebesgue sets $\mathcal{L}^k \subset \mathfrak{P}(\mathbb{R}^k)$ will be generated?
11. Construct for an arbitrary k-dimensional Lebesgue null set $\mathrm{N}_0 \subset \mathbb{R}^k$ a Borel set $\mathrm{N} \supset \mathrm{N}_0$, which is a k-dimensional Lebesgue null set as well.
12. Describe the Cantor set $\mathrm{C} \subset \mathbb{R}^1$ and name its most important topological and algebraical properties.
13. Explain: atomic and diffuse measure.
14. State the theorem about continuity from below, from above and at zero for a measure μ.
15. What is a Dynkin system, and when is a Dynkin system a σ-algebra?
16. In relation to an outer measure $\mu\colon \mathfrak{P}(\Omega) \to \overline{\mathbb{R}}$, describe the system of μ-measurable sets and its properties.
17. State the extension theorem for a measure μ defined on a ring $\mathcal{A} \subset \mathfrak{P}(\Omega)$.
18. When a σ-finite measure $\mu\colon \mathcal{B}^k \to \overline{\mathbb{R}}$ is called regular?
19. Characterize absolute continuity $\mu \ll \lambda$ by an ε-δ-relation.
20. Describe the three decomposition parts of an arbitrary σ-finite measure μ on $[\,\mathbb{R}^k, \mathcal{B}^k\,]$.

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“Measure and Integration” — Examination Questions (II)

21. When a finite content $\mu\colon \mathcal{A} \to \mathbb{R}$, $\mu \neq \mathfrak{o}$, is called purely finitely additive?
22. What is an $\mathcal{A}$-$\mathcal{B}^1$-measurable function?
23. Prove that the pointwise maximum $\mathrm{Max}(f, g)$ of two $\mathcal{A}$-$\mathcal{B}^1$-measurable functions f, g is $\mathcal{A}$-$\mathcal{B}^1$-measurable as well.
24. State Lusin's theorem.
25. Explain: Carathéodory function.
26. In relation to a measure space $[\,\Omega\,,\,\mathcal{A}\,,\,\mu\,]$, define the set of nonnegative step functions $T^+(\Omega)$ and the measure integral $\int_\Omega \varphi(x)\,d\mu(x)$ for $\varphi \in T^+(\Omega)$.
27. Define the measure integral for a nonnegative, $\mathcal{A}$-$\overline{\mathcal{B}}^1$-measurable function f and explain how the definition is justified.
28. Prove that an $\mathcal{A}$-$\overline{\mathcal{B}}^1$-measurable function f is μ-integrable iff $|\,f\,|$ is μ-integrable.
29. State Fatou's lemma in its extended form.
30. How the Dominated Convergence theorem reads for $f(x) = \lim_{n\to\infty} f_n(x)$?
31. How the Dominated Convergence theorem reads for $f(x) = \sum_{n=1}^{\infty} f_n(x)$?
32. Express the Lebesgue integral $L\text{-}\int_a^b f(x)\,dx$ as the limit of Lebesgue sums.
33. Under what conditions the Lebesgue integral $L\text{-}\int_0^\infty f(x)\,dx$ and the improper Riemann integral $R\text{-}\int_0^\infty f(x)\,dx$ coincide?
34. For a Lipschitz function $f\colon\ [\,a\,,\,b\,] \to \mathbb{R}$, express the difference $f(x'') - f(x')$ as a Lebesgue integral.
35. With respect to the k-dimensional Lebesgue measure, define the essential infimum $\operatorname{ess\,inf}_{x\in\Omega} f(x)$ and the essential supremum $\operatorname{ess\,sup}_{x\in\Omega} f(x)$ for a Borel-measurable function $f\colon\ \Omega \to \mathbb{R}$.
36. Let $\Omega \subset \mathbb{R}^k$ be the closure of a bounded domain. Explain the relations between the spaces $L^p(\Omega, \lambda^k)$, $1 \leqslant p \leqslant \infty$.
37. State the mean value theorem for the measure integral.
38. How the definition of a null set must be modified for a signed measure ν on the measurable space $[\,\Omega\,,\,\mathcal{A}\,]$?
39. State the Hahn decomposition theorem on a measure space $[\,\Omega\,,\,\mathcal{A}\,,\,\nu\,]$ with a signed measure ν.
40. For a signed measure ν on the measurable space $[\,\Omega\,,\,\mathcal{A}\,]$, define its positive part ν^+, its negative part ν^- and the total variation $\vee\,\nu$.

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“Measure and Integration” — Examination Questions (III)

41. Show that $\vee\kappa \ll \kappa$ for every signed measure.

42. State the Radon-Nikodym theorem for signed measures.

43. Prove the triangle inequality $\| \nu' + \nu'' \| \leqslant \| \nu' \| + \| \nu'' \|$ for signed measures ν', ν'' on the measurable space $[\, \Omega \,,\, \mathcal{A} \,]$.

44. Define the normed spaces $ca\,[\, \Omega \,,\, \mathcal{A} \,]$ and $rca\,[\, \Omega \,,\, \mathcal{B}_\Omega^K \,]$.

45. Explain how the measure integral can be defined with respect to a signed measure ν on $[\, \Omega \,,\, \mathcal{A} \,]$.

46. How the Riesz representation theorem for $\big(C^0(\Omega) \big)^*$ reads?

47. Describe the dual space $\big(L^\infty[\, \Omega \,,\, \mathcal{B}_\Omega^k \,,\, \lambda^k \,] \big)^*$.

48. Define the weak derivative $g = Df$ of a function $f\colon\ [\, a \,,\, b \,] \to \mathbb{R}$.

49. What is a Schwartz distribution?

50. Describe the space $BV(\Omega)$ of the functions of bounded variation and the norm used on it.