

Course “Calculus of Variations” (winter term 2003/04) — Contents

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9. Second-order necessary and sufficient conditions for the basic problem (II)
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